Index options: a model-free approach.

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Abstract

This paper contains an overview and an extension of the theory on comonotonicity-based model-free upper bounds and super-replicating strategies for stock index options, as presented in Hobson et al. (2005) and Chen et al. (2008). Whereas these authors only consider index call options, here a unified approach for call and put options is presented. Considering a unified framework gives rise to an efficient algorithm for calculating upper bounds and for determining the corresponding super-hedging strategies for both cases. The unified framework also allows to extend several existing results, in particular on the optimality of the superhedging strategies. Several practical issues concerning the implementation of the results are discussed. In particular, a simplified algorithm is presented for the situation where for some of the constituent stock in the index there are no options available.

Keywords: index call and put options, comonotonicity, model-free approach, static super-replicating strategies.

1 Introduction

In this paper we investigate European-type options on an index which is a weighted sum of stock prices. The usual setup for determining the arbitrage-free price of such an option consists of first postulating a risk-neutral measure and then determining its price as the expected value of its discounted pay-off, where discounting is performed at the risk-free rate and the expectation is taken with respect to the risk-neutral measure. We will consider a different approach. Instead of postulating a risk-neutral measure, we will look for the best upper bound for the price of the index option under consideration, based

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on available market information. In particular, we will determine the lowest upper bound for the price of the index option which is consistent with the observed prices of traded European options on the individual stocks contained in the index. We will prove that this upper bound corresponds to the price of the cheapest strategy in a broad class of static investment strategies with a pay-off that super-replicates the pay-off of the index option.

We first consider the *infinite market case*, where the prices of the options on the stocks of which the index is composed are available for all strikes. We prove that in this case, the cheapest super-replicating strategy for the index option consists of buying for each individual stock only one type of option on that stock. Armed with the results of the infinite market case, we are able to investigate the more realistic *finite market case*, where only a finite number of options on each individual stock are traded. In the finite market case, it turns out that the cheapest super-replicating strategy consists of buying for each individual stock options with at most two different strike prices.

The approach followed in this chapter is a *model-free approach* in the sense that the upper bound for the index option price and the corresponding super-hedging strategy are determined from the observed option prices on the individual stocks, without making any assumption about the underlying risk-neutral measure.

This paper is of a pedagogical nature. It is closely related to earlier work of Hobson et al. (2005) and Chen et al. (2008). In order to make this paper self-contained, we repeat their results on index call options. Furthermore, we develop corresponding results for index put options. Considering the pricing of index call and put options in a unified framework gives rise to an efficient algorithm for calculating upper bounds and for determining the corresponding superhedging strategies for both cases. The unified framework also allows us to extend existing optimality results concerning these superhedging strategies. We also consider the situation where for some of the constituent stocks in the index there are no options available. We show how the algorithm for calculating bounds and super-replicating strategies can be further simplified in this case. One of the aims of our paper is to make this extended version of the work of Hobson et al. (2005) and Chen et al. (2008) accessible to a broader audience by simplifying the original proofs and presentations and by considering several practical aspects concerning the implementation of these results. Based on the results presented in this paper, Dhaene et al. (2011) and Dhaene et al. (2012) propose an easy to calculate measure for the implied degree of co-movement behavior in stock markets.

2 Stocks, the market index and options

Throughout this chapter, we assume a financial market[1] where n different (dividend or non-dividend paying) stocks, labeled from 1 to n, are traded. Current time is 0, while

[1] We use the common approach to describe the financial market via a filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P}\right)$, which satisfies the usual technical conditions of completeness and right-continuity, and where $\mathcal{F}_0$ contains all $\mathbb{P}$ - null sets of Ω. Price processes of traded financial instruments are modeled as stochastic processes on that probability space which are adapted to the filtration $(\mathcal{F}_t)_{0\leq t\leq T}$.

the time span under consideration is T years. The price of stock i at time t, $0 \leq t \leq T$, is denoted by the non-negative random variable (r.v.) $X_i(t)$. The stochastic price process of stock i is denoted by $\{X_i(t) \mid 0 \leq t \leq T\}$.

In this market, there is a market index which is a linear combination of the n traded stocks. Denoting the price of this index at time t by $S(t)$, $0 \leq t \leq T$, we have that

$$S(t) = w_1 X_1(t) + w_2 X_2(t) + \ldots + w_n X_n(t), \tag{1}$$

where w_i, $i = 1, 2, \ldots, n$, are positive weights that are fixed up front.

We assume that market participants have access to a number of European options with maturity T. More precisely, they can trade in European calls and puts on the index as well as on the individual stocks. We recall that the pay-off at time T of a European call with maturity T and strike K on the index is given by $(S(T) - K)_+$, whereas the pay-off of the corresponding index put option is given by $(K - S(T))_+$. The time-0 prices of these index options are denoted by $C[K,T]$ and $P[K,T]$, respectively. Similar pay-offs and notations hold for calls and puts on the constituent stocks. In particular, the time-0 prices of calls and puts on stock i are denoted by $C_i[K,T]$ and $P_i[K,T]$, respectively.

It is assumed that the financial market is arbitrage-free and that there exists a pricing measure $\mathbb{Q}$, equivalent to the physical probability measure $\mathbb{P}$, such that the current price of any pay-off at time T can be represented as the expectation of the discounted pay-off. In this price-recipe, the discounting factor is e^{-rT}, where r is the continuously compounded time-0 risk-free interest rate to expiration T, whereas expectations are taken with respect to $\mathbb{Q}$. For simplicity in notation and terminology, we assume deterministic interest rates. Notice however that all results hereafter remain to hold in case interest rates are stochastic, provided the discounting factor e^{-rT} is interpreted as the time-0 price of a T-year zero coupon bond and the pricing measure $\mathbb{Q}$ is interpreted as a 'T-year forward measure' instead of a 'risk-neutral measure'.

The no-arbitrage condition gives rise to the following expressions for the European call and put option prices:

$$C_i[K,T] = e^{-rT}\mathbb{E}[(X_i(T) - K)_+], \tag{2}$$
$$P_i[K,T] = e^{-rT}\mathbb{E}[(K - X_i(T))_+], \tag{3}$$

and

$$C[K,T] = e^{-rT}\mathbb{E}[(S(T) - K)_+], \tag{4}$$
$$P[K,T] = e^{-rT}\mathbb{E}[(K - S(T))_+]. \tag{5}$$

In formulae (2), (3), (4) and (5) as well as in the remainder of this text, expectations of functions of $(X_1(T), \ldots, X_n(T))$ have to be understood as expectations under the $\mathbb{Q}$-measure. We call such expectations risk-neutral expectations. Furthermore, the notations $F_{X_i(T)}(x)$ and $F_{S(T)}$ will be used for the time-0 cumulative distribution functions (cdf's) of $X_i(T)$ and $S(T)$ under $\mathbb{Q}$. We will call $F_{X_i(T)}(x)$ and $F_{S(T)}$ the risk-neutral distributions of the stock and index prices at time T, respectively.

In order to avoid unnecessary overloading of the notations, from here on we will omit the time index T when no confusion is possible. This means e.g. that we will use the notations X_i, $C_i[K]$ and $F_{X_i}(x)$ for $X_i(T)$, $C_i[K,T]$ and $F_{X_i(T)}(x)$, respectively.

One of the goals of this chapter is to determine the smallest upper bound for the index call and put option prices $C[K]$ and $P[K]$ which can be expressed as the price of a portfolio of individual stocks and options on these individual stocks, with a pay-off that super-replicates the pay-off of the index option under consideration. Solving this problem numerically by considering any feasible combination of stock options is practically impossible. Indeed, let us consider the simpler problem where we want to determine the lowest upper bound for the call index option price $C[K]$ which can be expressed as the price $\sum_{i=1}^{n} w_i C_i[K_i]$ of a super-replicating strategy for this index call option consisting of buying for each stock i in the index a number of w_i call options $C_i[K_i]$ with strike K_i. From a practical point of view, solving this problem numerically by considering any feasible vector $(K_1, K_2, \ldots, K_n)$ of available strikes is impossible. Assuming that the number of traded strikes per vanilla option is equal to m, this problem comes down to finding the price of the cheapest super-replicating strategy among a set of m^n possible combinations. In case of the Dow Jones Index (DJI), which has 30 stocks in the index and an average number of around 10 traded strikes per individual stock, the number of possible combinations is of the order 10^{30}. The problem that we want to solve in this chapter is even much more complex, in the sense that we will not restrict to super-replicating strategies consisting of only one strike per stock. Instead, we will consider super-replicating strategies which allow to buy stock options for any traded strike. This example, which is described in Hobson et al. (2005), clearly illustrates the need for deriving an anaytical solution to the above-mentioned super-replication problem.

3 Convex order, inverse distributions and comonotonicity

In this section we summarize some definitions and results concerning convex order, inverse distributions and comonotonicity that will be needed in later sections. All random variables are assumed to have finite means.

A r.v. X is said to precede a r.v. Y in *convex order sense*, notation $X \leq_{\text{cx}} Y$, if

$$\begin{cases} \mathbb{E}\left[(X-K)_+\right] \leq \mathbb{E}\left[(Y-K)_+\right] \\ \mathbb{E}\left[(K-X)_+\right] \leq \mathbb{E}\left[(K-Y)_+\right] \end{cases}, \qquad \text{for all } K \in \mathbb{R}. \tag{6}$$

From (6) it is clear that $X \leq_{\text{cx}} Y$ intuitively means that Y has larger (upper and lower) tails than X.

The usual inverse F_X^{-1} of the cdf F_X of a r.v. X is defined by

$$F_{X_i}^{-1}(p) = \inf\left\{x \in \mathbb{R} \mid F_{X_i}(x) \geq p\right\}, \qquad p \in [0,1], \tag{7}$$

with $\inf \emptyset = +\infty$, by convention. For any $x \in \mathbb{R}$ and $p \in [0,1]$, the following equivalence relation holds:

$$F_X^{-1}(p) \leq x \Longleftrightarrow p \leq F_X(x). \tag{8}$$

An alternative definition for the inverse distribution function of F_X is given by:

$$F_X^{-1+}(p) = \sup \{x \in \mathbb{R} \mid F_{X_i}(x) \leq p\}, \qquad p \in [0,1], \tag{9}$$

with $\sup \emptyset = -\infty$, by convention. Both inverses (7) and (9) only differ on horizontal segments of the distribution function F_X. The interval $\left[F_X^{-1+}(0), F_X^{-1}(1)\right]$ is a support of X.
For any number $\alpha \in [0,1]$, the alpha inverse $F_X^{-1(\alpha)}$ is defined as a linear combination of (7) and (9):

$$F_X^{-1(\alpha)}(p) = \alpha F_X^{-1}(p) + (1-\alpha) F_X^{-1+}(p), \qquad p \in (0,1). \tag{10}$$

The random vector $(Y_1, \ldots, Y_n)$ is said to be *comonotonic* if

$$(Y_1, \ldots, Y_n) \stackrel{\mathrm{d}}{=} \left(F_{Y_1}^{-1}(U), \ldots, F_{Y_n}^{-1}(U)\right), \tag{11}$$

where U is a uniform $(0,1)$ r.v. and ‘ $\stackrel{\mathrm{d}}{=}$ ’ is used to denote ‘equality in distribution’.

Consider the random vector $(X_1, \ldots, X_n)$ and the positive weights $w_i > 0$. The weighted sum S is defined by

$$S = \sum_{i=1}^{n} w_i X_i.$$

The *comonotonic modification* S^c of the weighted sum S is defined by

$$S^c = w_1 F_{X_1}^{-1}(U) + w_2 F_{X_2}^{-1}(U) + \ldots + w_n F_{X_n}^{-1}(U). \tag{12}$$

Taking into account that

$$F_{X_i}^{-1}(U) \stackrel{\mathrm{d}}{=} X_i, \qquad i = 1, 2, \ldots, n, \tag{13}$$

we immediately find that

$$\mathbb{E}[S] = \mathbb{E}[S^c]. \tag{14}$$

Furthermore, the comonotonic sun is always larger in convex order than the sum S:

$$S \leq_{\text{cx}} S^c. \tag{15}$$

The convex order inequality (15) can be generalized as follows:

$$X_i \leq_{\text{cx}} Y_i \text{ for } i = 1, \ldots, n \Rightarrow \sum_{i=1}^{n} w_i X_i \leq_{\text{cx}} \sum_{i=1}^{n} w_i F_{Y_i}^{-1}(U). \tag{16}$$

For any $\alpha \in [0,1]$, the inverse distribution function $F_{S^c}^{-1(\alpha)}$ of a comonotonic sum can be expressed in terms of the marginal inverse distribution functions $F_{X_i}^{-1(\alpha)}$, $i = 1, 2, \ldots, n$:

$$F_{S^c}^{-1(\alpha)}(p) = \sum_{i=1}^{n} w_i F_{X_i}^{-1(\alpha)}(p), \qquad p \in (0,1). \tag{17}$$

For $K \in \left(F_{S^c}^{-1+}(0), F_{S^c}^{-1}(1)\right)$, the stop-loss premium $\mathbb{E}\left[(S^c - K)_+\right]$ of the comonotonic sum S^c can be decomposed into a linear combination of stop-loss premiums of the marginals involved:

$$\mathbb{E}\left[(S^c - K)_+\right] = \sum_{i=1}^{n} w_i \mathbb{E}\left[(X_i - K_i^*)_+\right], \tag{18}$$

where

$$K_i^* = F_{X_i}^{-1(\alpha_K)}\left(F_{S^c}(K)\right), \qquad i = 1, \ldots, n, \tag{19}$$

and α_K is any element in $[0, 1]$ that satisfies the following relation:

$$\sum w_i K_i^* = K. \tag{20}$$

For an extensive overview of the theory of comonotonicity, including proofs of the results mentioned in this subsection, we refer to Dhaene et al. (2002a). Financial and actuarial applications of the concept of comonotonicity are described in Dhaene et al. (2002b). An updated overview of applications of comonotonicity can be found in Deelstra et al. (2010).

4 The infinite market case

4.1 From option prices to risk-neutral distributions

In this section, we consider the situation where for each stock i, the prices $C_i[K]$ and $P_i[K]$ of the stock options are known for any strike $K \geq 0$. For obvious reasons, we call this situation the *infinite market case.* All these option prices are known because we either assume that any strike is traded so that the price of any put and call is observed in the market, or we assume that $\mathbb{Q}$ is known. The first approach is called *model-free* as it is based on the observed stock option prices, without making any assumption concerning the pricing measure $\mathbb{Q}$ that is actually used by the market. The second approach is called *model-based*, as it is based on a particular stock price model, such as the Black & Scholes model e.g.

From (2) and (3) it follows that

$$C_i[K] + e^{-rT}K = P_i[K] + e^{-rT}\mathbb{E}[X_i]. \tag{21}$$

This relation between the call and the put option prices with the same strike and maturity is known as the *put-call parity.* The term $e^{-rT}\mathbb{E}[X_i]$ can be interpreted as the zero-strike call option price:

$$C_i[0] = e^{-rT}\mathbb{E}[X_i]. \tag{22}$$

In case it is known that stock i will pay no dividends in $[0, T]$, we have that $C_i[0] = X_i(0)$. In general however, one has that

$$C_i[0] \leq X_i(0). \tag{23}$$

The put-call parity (21) can also be proven via a no-arbitrage argument. Indeed, consider the time zero strategy consisting of buying $C_i[K_{i,j}]$ and investing $K_{i,j}e^{-rT}$ in the risk-free account. The pay-off at time T of this strategy is equal to the pay-off at time T of the time zero strategy consisting of buying the options $P_i[K_{i,j}]$ and $C_i[0]$. Given that both strategies have the same pay-off at time T, they must have the same price at time 0.

The risk-neutral expectation $\mathbb{E}[X_i]$ in (21) can also be interpreted as the time-0 forward price of stock i at time T. Indeed, consider the contract set up at time 0, of which the buyer pays the stock price X_i at time T, while the seller in return pays a fixed amount P at time T, which was agreed upon at the deal's inception. Assuming that P is determined such that the price of the contract is 0 at time 0, i.e.

$$0 = e^{-rT}\mathbb{E}[X_i - P], \tag{24}$$

leads to the following expression for P:

$$P = \mathbb{E}[X_i]. \tag{25}$$

This contract is called a T-year forward contract on stock i, while $\mathbb{E}[X_i]$ is called the time-0 forward price of stock i at time T.

The put-call parity (21) with $K = 0$ connects the time-0 call option price $C_i[0]$ and the forward price $\mathbb{E}[X_i]$ with the prices of call and put options on stock i.

The risk-neutral distribution function F_{X_i} of X_i can be determined from the corresponding call option curve by the following equation:

$$F_{X_i}(x) = 1 + e^{rT}C_i'[x+], \tag{26}$$

where $C_i'[x+]$ is the right derivative of C_i at x; see e.g. Breeden and Litzenberger (1978). Using the put-call parity, it follows that F_{X_i} can also be derived from the corresponding put option curve:

$$F_{X_i}(x) = e^{rT}P_i'[x+]. \tag{27}$$

Given the call or the put option curve, the risk-neutral marginal distribution function F_{X_i} is fully determined. However, the observed stock option prices do not allow us to specify the multivariate pricing distribution $F_{X_1,X_2,\dots,X_n}(x_1, x_2, \dots, x_n)$.

In practice, it will never be the case that stock options are traded for all $K \geq 0$. Instead, only at most a finite number of such options will be traded per individual stock. This more realistic situation will be investigated in the next section, where we will consider the *finite market case*. However, we will consider the infinite market case first as the results for the finite market case will follow rather straightforward from transforming the finite market in an (artificial) infinite market. Furthermore, the results for the infinite market case presented in this section may be useful in a model-based approach, where a specific pricing measure $\mathbb{Q}$ is assumed. In this case the multivariate pricing distribution of the random vector $X \equiv (X_1, X_2, \dots, X_n)$ is specified. Nevertheless, determining the price of the index option analytically in this situation is in most cases still not a straightforward exercise, mainly because of the dependence that exists between the stock prices X_i, $i = 1, 2, \ldots, n$. Even in a Black & Scholes setting where the stock prices are driven by correlated geometric Brownian motions and the stock option prices $C_i[K]$ and $P_i[K]$ can easily be obtained for all K, the index option prices $C[K]$ and $P[K]$ are difficult to evaluate analytically. Therefore, the use of an easy computable upper bound in terms of the stock option prices involved may also in a model-based approach be very helpful as an approximation for the real price of the index option.

4.2 Upper bounds for index option prices

As before, the notation S is used to denote the value of the index at time T. Hence,

$$S = w_1 X_1 + w_2 X_2 + \ldots + w_n X_n, \tag{28}$$

where w_i, $i = 1, 2, \ldots, n$, are positive weights that are fixed up front. The comonotonic modification S^c of S is defined by

$$S^c = w_1 F_{X_1}^{-1}(U) + w_2 F_{X_2}^{-1}(U) + \ldots + w_n F_{X_n}^{-1}(U), \tag{29}$$

where U is a uniform $(0,1)$ random variable. We will call S^c the comonotonic index price at time T.

In practice, we will never observe the outcome of S^c, unless all stock prices $(X_1, X_2, \ldots, X_n)$ are comonotonic. Our goal is to find reasonable upper bounds for the index option prices $C[K]$ and $P[K]$ which can be expressed in terms of the information contained in the observed stock option prices. We start our search for such bounds by deriving upper bounds for $C[K]$ and $P[K]$ in terms of the cdf of S^c.

Theorem 1 (Upper bounds for index option prices) *The prices $C[K]$ and $P[K]$ of the index options with pay-off at time T given by $(S-K)_+$ and $(K-S)_+$, respectively, are constrained from above as follows:*

$$C[K] \leq \mathrm{e}^{-rT}\mathbb{E}\left[(S^c - K)_+\right], \tag{30}$$

$$P[K] \leq \mathrm{e}^{-rT}\mathbb{E}\left[(K - S^c)_+\right]. \tag{31}$$

Proof. The inequalities (30) and (31) follow immediately from the characterization (6) of the convex order relation (15) and by taking into account the expressions (4) and (5) for index call and put options prices. ■

The right hand sides of (30) and (31) correspond to the prices of an index call and put option with strike K in case the dependence structure is the comonotonic one. In the sequel, we will denote these prices by $C^c[K]$ and $P^c[K]$, respectively:

$$C^c[K] = e^{-rT}\mathbb{E}\left[(S^c - K)_+\right], \tag{32}$$

$$P^c[K] = e^{-rT}\mathbb{E}\left[(K - S^c)_+\right], \tag{33}$$

and call them the *comonotonic call and put option prices.*

For $K \notin \left(F_{S^c}^{-1+}(0), F_{S^c}^{-1}(1)\right)$, we know the exact values of the index option prices $C[K]$ and $P[K]$. Indeed, it is straightforward to verify that

$$C[K] = \begin{cases} e^{-rT}\left(\mathbb{E}[S] - K\right), & K \leq F_{S^c}^{-1+}(0), \\ 0, & K \geq F_{S^c}^{-1}(1), \end{cases} \tag{34}$$

while

$$P[K] = \begin{cases} 0, & K \leq F_{S^c}^{-1+}(0), \\ e^{-rT}\left(K - \mathbb{E}[S]\right), & K \geq F_{S^c}^{-1}(1). \end{cases} \tag{35}$$

In these expressions, $e^{-rT}\mathbb{E}[S]$ is equal to the zero-strike index call option price $C[0]$. It can be determined from the zero-strike stock option prices:

$$e^{-rT}\mathbb{E}[S] = \sum_{i=1}^{n} w_i C_i[0]. \tag{36}$$

The quantity $\mathbb{E}[S]$ can be interpreted as the time-0 forward price of the market index at time T. From the put-call parity

$$C[K] + e^{-rT}K = P[K] + e^{-rT}\mathbb{E}[S] \tag{37}$$

for index options, it follows that $\mathbb{E}[S]$ can also be determined from observed index call and put option prices. Notice that for the comonotonic index and its related comonotonic option prices, the following put-call parity holds:

$$C^c[K] + e^{-rT}K = P^c[K] + e^{-rT}\mathbb{E}[S]. \tag{38}$$

It is straightforward to prove that

$$C[K] = C^c[K] \text{ and } P[K] = P^c[K], \qquad \text{if } K \notin \left(F_{S^c}^{-1+}(0), F_{S^c}^{-1}(1)\right). \tag{39}$$

This means that the upper bounds in Theorem 1 coincide with the exact option prices in this case.

As the values of $C[K]$ and $P[K]$ are explicitely known for $K \notin \left(F_{S^c}^{-1+}(0), F_{S^c}^{-1}(1)\right)$, in the sequel we will focus on the case where $K \in \left(F_{S^c}^{-1+}(0), F_{S^c}^{-1}(1)\right)$ when considering upper bounds for index option prices.

In the following theorem, we show that both upper bounds for index options that were derived in Theorem 1 can be expressed as a linear combination (l.c.) of observed stock option prices.

Theorem 2 (C^c and P^c are l.c.'s of stock option prices) *For any $K \in \left(F_{S^c}^{-1+}(0), F_{S^c}^{-1}(1)\right)$ the comonotonic index option prices $C^c[K]$ and $P^c[K]$ can be expressed as*

$$C^c[K] = \sum_{i=1}^{n} w_i C_i[K_i^*], \tag{40}$$

$$P^c[K] = \sum_{i=1}^{n} w_i P_i[K_i^*], \tag{41}$$

with the K_i^* *given by*

$$K_i^* = F_{X_i}^{-1(\alpha_K)}\left(F_{S^c}(K)\right), \qquad i = 1, 2, \ldots, n \tag{42}$$

and where α_K *is any element in* $[0,1]$ *such that*

$$\sum_{i=1}^{n} w_i K_i^* = K. \tag{43}$$

Proof. Taking into account expression (2) for the stock option curve C_i, we can rewrite the decomposition formula (18) as follows:

$$C^c[K] = \sum_{i=1}^{n} w_i C_i[K_i^*], \tag{44}$$

which proves (40). Using the put-call parities (21) and (38), one can transform (44) into

$$P^c[K] + e^{-rT}\mathbb{E}[S] - e^{rT}K = \sum_{i=1}^{n} w_i \left(P_i[K_i^*] + e^{-rT}\mathbb{E}[X_i] - e^{rT}K_i^*\right).$$

Combining this expression with (43) proves assertion (41). ■

From the additivity property (17) of quantiles of a comonotonic sum, it follows that relation (43) which is used for determining α_K can be rewritten as

$$F_{S^c}^{-1(\alpha)}\left(F_{S^c}(K)\right) = K. \tag{45}$$

In order to be able to calculate the optimal strike prices K_i^* one has to determine $F_{S^c}(K)$ and α_K. The determination of these quantities is considered in Section 4.5.

The comonotonic call option price $C^c[K]$ corresponds to the price of a super-replicating strategy for the index call option with pay-off $(S-K)_+$ at time T, whereas the comonotonic put option price $P^c[K]$ corresponds to the price of a super-replicating strategy for the index put option with pay-off $(K-S)_+$ at time T. These statements are proven in the following theorem.

Theorem 3 (C^c and P^c are the prices of static super-replicating strategies) *Let* $K \in \left(F_{S^c}^{-1+}(0), F_{S^c}^{-1}(1)\right)$ *and consider the index call and put options with pay-off at time* T *given by* $(S-K)_+$ *and* $(K-S)_+$*, respectively.*

1. *The pay-off of the static strategy where at time* 0*, for each stock* i*,* $i \in \{1, 2, \ldots, n\}$*, one buys* w_i *calls* $C_i[K_i^*]$ *and holds these positions until they expire at time* T*, super-replicates the pay-off of the index call option with price* $C[K]$*. The price of this super-replicating strategy is given by* $C^c[K]$*.*

2. *The pay-off of the the static strategy where at time* 0*, for each stock* i*,* $i \in \{1, 2, \ldots, n\}$*, one buys* w_i *puts* $P_i[K_i^*]$ *and holds these positions until they expire at time* T*, super-replicates the pay-off of the index put option with price* $P[K]$*. The price of this super-replicating strategy is given by* $P^c[K]$*.*

Proof. The pay-off at time T of the index option $C[K]$ is given by $\left(\sum_{i=1}^n w_i X_i - K\right)_+$, while the pay-off at time T of the time-0 strategy consisting of buying w_i call options $C_i[K_i^*]$ and holding these options until maturity is given by $\sum_{i=1}^n w_i(X_i - K_i^*)_+$. As $\sum_{i=1}^n w_i K_i^* = K$, we have that the following inequality holds:

$$\left(\sum_{i=1}^n w_i X_i - K\right)_+ \leq \sum_{i=1}^n w_i(X_i - K_i^*)_+, \tag{46}$$

which proves that the pay-off of this time-0 strategy super-replicates the pay-off of the index call option. Obviously, the price of this strategy is given by $\sum_{i=1}^n w_i C_i[K_i^*]$, which according to (40) is equal to $C^c[K]$.
From (46) one finds that

$$\left(K - \sum_{i=1}^n w_i X_i\right)_+ \leq \sum_{i=1}^n w_i(K_i^* - X_i)_+. \tag{47}$$

The left hand side of this inequality is the pay-off at time T of the index put option $P[K]$, whereas its right hand side equals the pay-off at time T of the time-0 strategy consisting of buying w_i vanilla put options $P_i[K_i^*]$ and holding these options until they expire at time T. Hence, this static time-0 strategy super-replicates the index put option pay-off $(K-S)_+$. The price of this super-replicating strategy is given by $\sum_{i=1}^n w_i P_i[K_i^*]$, which according to (41) is equal to $P^c[K]$. ∎

Theorem 3 shows that in the presence of traded call and put options on the constituent stocks of the index, an index call option can be superhedged with stock call options, while an index put option can be superhedged with stock put options. From this observation we find that the price inequalities

$$C[K] \leq \sum_{i=1}^n w_i C_i[K_i^*]$$

and

$$P[K] \leq \sum_{i=1}^n w_i P_i[K_i^*]$$

remain to hold, without having to make the explicit assumption that the involved option prices are expectations of discounted pay-offs under some $\mathbb{Q}$-measure. The only assumption that we have to make is that all option prices involved are traded prices in an arbitrage-free market, implying that a superhedging strategy for the index option is more expensive than the index option itself. Notice however that in order to prove the equalities (40) and (41), we have to assume that option prices can be expressed as expectations of their discounted pay-offs.

4.3 The upper bound is the price of the cheapest super-replicating strategy

The upper bounds that we derived for the index call and put option prices are linear combinations of n observed stock option prices. To be more precise, the linear combination contains w_i options on the underlying X_i with strike price K_i^*. The question arises whether it is possible to derive better upper bounds for the price of both types of index options within a general class of superhedging strategies consisting of buying or selling call and put options on the underlying stocks. In order to be able to answer this question, we first have to define this general class of superheding strategies. Hereafter, we use 'r.c.' as an abbreviation for 'right continuous'.

Definition 1 (The class $\mathcal{I}$) *The class $\mathcal{I}$ consists of all $2n$-dimensional functions $\underline{\nu} \equiv (\nu_{1c}, \nu_{1p}, \nu_{2c}, \nu_{2p} \ldots, \nu_{nc}, \nu_{np})$, of which for each i, the functions $\nu_{ic} : \mathbb{R} \to \mathbb{R}$ and $\nu_{ip} : \mathbb{R} \to \mathbb{R}$ are r.c. jump functions with $\nu_{ic}(y) = \nu_{ip}(y) = 0$ for any $y < 0$, and having only a finite number of jumps in $[0, +\infty)$. Jumps upwards as well as downwards are allowed.*

We will consider the class of investment strategies where for each stock i at current time 0, stock options can be bought (i.e. holding a long position) or sold (i.e. holding a short position) for any strike $y \geq 0$. The positions taken are assumed to be held until time T, and then eventually exercised. We describe any such investment strategy by a vector of functions $\underline{\nu} \in \mathcal{I}$, where for any stock i and any strike $y \geq 0$, we interprete $\nu_{ic}(y)$ as the number of call options purchased with a strike price smaller than or equal to y. Similarly, for any stock i and any strike $y \geq 0$, the value of $\nu_{ip}(y)$ is the number of put options purchased with a strike price smaller than or equal to y. Notice that selling a number of n options of a certain type can be expressed as buying $(-n)$ of these options. A jump upwards in one of the components of $\underline{\nu}$ corresponds to a long position, whereas a jump downwards corresponds to a short position. Although the assumption about the finite number of jumps can be relaxed, we will keep it here as it is a reasonable assumption which will always be met in real-life investment strategies where obviously only a finite number of strikes will be purchased per stock.

For each stock i, we use the symbol $J_{\nu_{ic}}$ to denote the finite set containing all values of y at which the function $\nu_{ic}(y)$ jumps, whereas $\Delta\nu_{ic}(y)$ is used to denote the magnitude of the jump at y:

$$\Delta\nu_{ic}(y) = \nu_{ic}(y) - \nu_{ic}(y-). \tag{48}$$

The notation $\nu_{ic}(y-)$ is used for the left limit $\lim_{\varepsilon\downarrow 0} \nu_{ic}(y-\varepsilon)$ at y. A positive value of $\Delta\nu_{ic}(y)$ means that an amount of $\Delta\nu_{ic}(y)$ call options $C_i[y]$ is purchased, whereas a negative value of $\Delta\nu_{ic}(y)$ corresponds with selling short an amount of $\Delta\nu_i(y)$ of these options. The functions $J_{\nu_{ip}}$ and $\Delta\nu_{ip}(y)$ are defined analogously.

The pay-off at time T of the investment strategy $\underline{\nu} \in \mathcal{I}$ is given by

$$\textsf{Pay-off}\ [\underline{\nu}, \underline{X}] = \sum_{i=1}^{n} \left(\sum_{y \in J_{\nu_{ic}}} (X_i - y)_+\ \Delta\nu_{ic}(y) + \sum_{y \in J_{\nu_{ip}}} (y - X_i)_+\ \Delta\nu_{ip}(y) \right), \tag{49}$$

where $\underline{X} \equiv (X_1, X_2, \ldots, X_n)$ is the vector of the individual stock prices at time T. The corresponding price of this investment strategy is given by

$$\textsf{Price}\ [\underline{\nu}] = \sum_{i=1}^{n} \left(\sum_{y \in J_{\nu_{ic}}} C_i[y]\ \ \Delta\nu_{ic}(y) + \sum_{y \in J_{\nu_{ip}}} P_i[y]\ \ \Delta\nu_{ip}(y) \right). \tag{50}$$

Introducing Riemann-Stieltjes integrals, we can rewrite the expressions for the pay-off and the price of the investment strategy $\underline{\nu} \in \mathcal{I}$ as follows:

$$\textsf{Pay-off}\ \left[\underline{\nu}, \underline{X}\right] = \sum_{i=1}^{n} \left(\int_{-\infty}^{+\infty} \left(X_i - y\right)_+ \ \mathrm{d}\nu_{ic}\left(y\right) + \int_{-\infty}^{+\infty} \left(y - X_i\right)_+ \ \mathrm{d}\nu_{ip}\left(y\right) \right) \tag{51}$$

and

$$\textsf{Price}\ \left[\underline{\nu}\right] = \sum_{i=1}^{n} \left(\int_{-\infty}^{+\infty} C_i\left[y\right] \ \mathrm{d}\nu_{ic}(y) + \int_{-\infty}^{+\infty} P_i\left[y\right] \ \mathrm{d}\nu_{ip}(y) \right). \tag{52}$$

Herafter, we will use the expressions (51) and (52) to denote pay-offs and prices of investment strategies in $\mathcal{I}$.

Example 1 (Two simple investment strategies) *The investment strategy*

$$\underline{\nu}^* \equiv \left(\nu_{1c}^*, \nu_{1p}^*, \ldots, \nu_{nc}^*, \nu_{np}^*\right) \in \mathcal{I}$$

is defined such that for $i = 1, 2, \ldots, n$, we have that

$$\nu_{ic}^*(y) = \begin{cases} 0 & y < K_i^*, \\ w_i & y \geq K_i^*, \end{cases} \quad \textit{and } \nu_{ip}^* \equiv 0,$$

with the K_i^* defined in (42). This strategy consists of buying w_i calls $C_i\left[K_i^*\right]$ for any stock i, whereas no put option is purchased. This investment strategy is the one that is considered in the first part of Theorem 3. Taking into account (40), we find that the price of the investment strategy $\underline{\nu}^*$ is given by

$$\textsf{Price}\ \left[\underline{\nu}^*\right] = \sum_{i=1}^{n} w_i C_i\left[K_i^*\right] = C^c\left[K\right].$$

At time T, this strategy will generate the following pay-off:

$$\textsf{Pay-off}\ \left[\underline{\nu}^*, \underline{X}\right] = \sum_{i=1}^{n} w_i (X_i - K_i^*)_+.$$

Similarly, we define the investment strategy

$$\underline{\eta}^* = \left(\eta_{1c}^*, \eta_{1p}^*, \ldots, \eta_{nc}^*, \eta_{np}^*\right) \in \mathcal{I}$$

by

$$\eta_{ic}^*(y) \equiv 0 \ \textit{and } \eta_{ip}^*(y) = \begin{cases} 0 & y < K_i^*, \\ w_i & y \geq K_i^*. \end{cases}$$

The investment strategy $\underline{\eta}^*$ is the one that was considered in the second part of Theorem 3. The price of $\underline{\eta}^*$ is given by

$$\textsf{Price}\ \left[\underline{\eta}^*\right] = \sum_{i=1}^{n} w_i P_i\left[K_i^*\right] = P^c\left[K\right],$$

while its pay-off at time T equals

$$\textit{Pay-off}\ \left[\underline{\eta}^*, \underline{X}\right] = \sum_{i=1}^{n} w_i (K_i^* - X_i)_+.$$

From Theorem 3 it follows that:

$$\begin{aligned}(S-K)_+ &\leq \textit{Pay-off}\ \left[\underline{\nu}^*, \underline{X}\right], \\ (K-S)_+ &\leq \textit{Pay-off}\ \left[\underline{\eta}^*, \underline{X}\right],\end{aligned}$$

which means that the strategy $\underline{\nu}^*$ is a super-replicating strategy for the index call option with pay-off $(S-K)_+$, whereas $\underline{\eta}^*$ is a super-replicating strategy for the index put option with pay-off $(K-S)_+$. ∇

The question arises whether it is possible to find better super-replicating strategies for the index options than the one that were considered in the previous example. Hence, can we find other strategies $\underline{\nu} \in \mathcal{I}$ which super-replicate the pay-off of the corresponding index option but are cheaper than $\underline{\nu}^*$, resp. $\underline{\eta}^*$? In order to be able to answer this question, we first have to define the subsets $\mathcal{C}_K$ and $\mathcal{P}_K$ of $\mathcal{I}$, containing all super-replicating strategies for the index call option $C[K]$ and put option $P[K]$, respectively. Obviously, $\underline{\nu}^* \in \mathcal{C}_K$ and $\underline{\eta}^* \in \mathcal{P}_K$, but are they the cheapest element in their respective classes?

Definition 2 (The classes $\mathcal{C}_K$ and $\mathcal{P}_K$) *For any $K \geq 0$, the classses $\mathcal{C}_K$ and $\mathcal{P}_K$ are defined by*

$$\mathcal{C}_K = \left\{ \underline{\nu} \in \mathcal{I} \mid \left(\sum_{i=1}^{n} w_i\ x_i - K \right)_+ \leq \textit{Pay-off}\ \left[\underline{\nu}, \underline{x}\right]\ \textit{for all}\ \underline{x} \right\}$$

and

$$\mathcal{P}_K = \left\{ \underline{\nu} \in \mathcal{I} \mid \left(K - \sum_{i=1}^{n} w_i\ x_i \right)_+ \leq \textit{Pay-off}\ \left[\underline{\nu}, \underline{x}\right]\ \textit{for all}\ \underline{x} \right\},$$

respectively. In these definitions, $\underline{x} \equiv (x_1, x_2, \ldots, x_n)$ and 'for all $\underline{x}$' has to be interpreted as

'for all $\underline{x}$ with $x_i \in$ Support $[X_i]$, $i = 1, 2, \ldots, n$'.

From the assumptions we made concerning the infinite market case it follows that we know the supports of any stock price X_i in the $\mathbb{Q}$-world, and hence, also in the $\mathbb{P}$-world. The latter conclusion follows from the fact that $\mathbb{P}$ and $\mathbb{Q}$ are equivalent, which implies that they agree on sure events and hence, also on supports. The set of $\underline{x}$-values for which the inequalities in the definitions above have to hold is a support of $(X_1, X_2, \ldots, X_n)$. We can conclude that

$$\mathbb{P}\left[(S-K)_+ \leq \text{Pay-off}\ \left[\underline{\nu}, \underline{X}\right]\right] = 1, \quad \text{for any } \underline{\nu} \in \mathcal{C}_K, \tag{53}$$

which means that the pay-off of any investment strategy $\underline{\nu} \in \mathcal{C}_K$ almost surely super-replicates the pay-off of the index call option. A similar remark holds for the pay-off of any investment strategy $\underline{\nu} \in \mathcal{P}_K$.

Example 2 (Super-replicating strategies) *Consider the investment strategy*

$$\underline{\nu} \equiv (\nu_{1c}, \nu_{1p}, \ldots, \nu_{nc}, \nu_{np}) \in \mathcal{I}$$

where for $i = 1, 2, \ldots, n$, the functions ν_{ic} and ν_{ip} are given by

$$\nu_{ic}(y) = \begin{cases} 0 & y < K_i, \\ w_i & y \geq K_i, \end{cases} \quad \text{and } \nu_{ip} \equiv 0,$$

and where the $K_i \geq 0$ are such that they satifsy

$$\sum_{i=1}^{n} w_i K_i \leq K. \tag{54}$$

By a triangle inequality, one can prove that condition (54) leads to

$$\left(\sum_{i=1}^{n} w_i x_i - K \right)_+ \leq \sum_{i=1}^{n} w_i \left(x_i - K_i \right)_+ = \textsf{Pay-off}\ \left[\underline{\nu}, \underline{x} \right],$$

which holds for any $\underline{x}$. Hence we can conclude that $\underline{\nu}$ belongs to $\mathcal{C}_K$. In particular, we find from (43) that the investment strategy $\underline{\nu}^*$ defined in Example 1 belongs to the set $\mathcal{C}_K$. ∇

Now we are equipped with the tools required for finding the cheapest super-replicating investment strategy for the index call and put options $C\left[K\right]$ and $P\left[K\right]$, respectively.

Theorem 4 (The price of the cheapest super-replicating strategy) *Let $\underline{\nu}^* \in \mathcal{C}_K$ and $\underline{\eta}^* \in \mathcal{P}_K$ be the investment strategies defined in Example 1.*
For any $K \in \left(F_{S^c}^{-1+}(0), F_{S^c}^{-1}(1) \right)$ it holds that

$$\min_{\underline{\nu} \in \mathcal{C}_K} \textsf{Price}\ \left[\underline{\nu} \right] = \textsf{Price}\ \left[\underline{\nu}^* \right] = C^c \left[K \right] \tag{55}$$

and

$$\min_{\underline{\nu} \in \mathcal{P}_K} \textsf{Price}\ \left[\underline{\nu} \right] = \textsf{Price}\ \left[\underline{\eta}^* \right] = P^c \left[K \right]. \tag{56}$$

Proof. Consider the super-replicating investment strategy $\underline{\nu} \in \mathcal{C}_K$. Replacing the x_i by $F_{X_i}^{-1}(U)$ in the pay-off inequality

$$\left(\sum_{i=1}^{n} w_i x_i - K \right)_+ \leq \sum_{i=1}^{n} \left(\int_{-\infty}^{+\infty} \left(X_i - y \right)_+ \ \mathrm{d}\nu_{ic}\left(y \right) + \int_{-\infty}^{+\infty} \left(y - X_i \right)_+ \ \mathrm{d}\nu_{ip}\left(y \right) \right)$$

and taking expectations leads to

$$\mathbb{E}\left[\left(S^c - K \right)_+ \right] \leq \sum_{i=1}^{n} \left(\int_{-\infty}^{+\infty} \mathbb{E}\left[(X_i - y)_+ \right] \ \mathrm{d}\nu_{ic}(y) + \int_{-\infty}^{+\infty} \mathbb{E}\left[(y - X_i)_+ \right] \ \mathrm{d}\nu_{ip}(y) \right).$$

Multiplying the left- and the right hand side by e^{-rT}, this inequality can be rewritten as

$$C^c[K] \leq \mathsf{Price}\ [\underline{\nu}].$$

As this inequality holds for any $\underline{\nu} \in \mathcal{C}_K$, we can conclude that

$$C^c[K] \leq \inf_{\underline{\nu} \in \mathcal{C}_K} \mathsf{Price}\ [\underline{\nu}].$$

On the other hand, as $\underline{\nu}^* \in \mathcal{C}_K$, we have that

$$\inf_{\underline{\nu} \in \mathcal{C}_K} \mathsf{Price}\ [\underline{\nu}] \leq \mathsf{Price}\ [\underline{\nu}^*] = C^c[K].$$

Combining these results, it follows that the stated results hold true for the call option case. The put option case can be proven in a similar way. ■

From Theorem 4, it is clear that the cheapest super-replicating strategy contained in $\mathcal{C}_K$, resp. $\mathcal{P}_K$, is the one that we considered in Example 1. The price of this cheapest super-replicating strategy is equal to the upper bound $C^c[K]$, resp. $P^c[K]$, that we derived in Theorem 1 for the index option price. Hence, we must answer 'no' to the question whether it is possible to improve the upper bounds derived in Theorem 1 by allowing for more than one type of option per individual stock. Although we allow portfolios consisting of an arbitrary number of calls and puts per stock, the cheapest of these strategies only invests in a single type of option (calls or puts) and a single strike per stock. This is a somewhat surprising result. Notice that this result does not mean that we limit the information used for deriving the upper bounds to a single option price per stock. Indeed, in order to determine the optimal strikes K_i^*, we also need the additional information see (42) in Theorem 2.

Theorem 4 can easily be generalized to the broader class of static super-replicating strategies which also contains investments in the risk-free account and in any contingent claim generating a pay-off $H(X_i)$ at time T provided that the time-0 price of this contingent claim is given by

$$\mathsf{Price}[H(X_i)] = e^{-rT}\mathbb{E}[H(X_i)]. \tag{57}$$

In this more general case, we simply have to redefine $\mathcal{I}$, $\mathcal{C}_K$ and $\mathcal{P}_K$ in terms of the available investment instruments, whereas the proof of the generalized optimisation result proceeds in the same way as the proof of Theorem 4.

Let us now suppose that neither the index call option $C[K]$ nor the index put option $P[K]$ is traded in the market. In case $C[K]$ is sold over-the-counter, then $C^c[K]$ may be a reasonable price for the index call option, both from the viewpoint of the seller and the buyer. Indeed, the seller can use this amount to acquire the portfolio $\underline{\nu}^*$, which will always super-replicate the pay-off of the index option that he is due to the buyer. On the other hand, the buyer of the index call option cannot find a cheaper super-replicating strategy in the market. In case the index option was sold over-the-counter at a higher price than the comonotonic price $C^c[K]$, the buyer may prefer to buy the cheaper super-replicating portfolio $\underline{\nu}^*$. A similar argument holds for the index put option that is sold over-the-counter.

4.4 The upper bound is the least upper bound for the index option price

We introduce the symbol $\mathcal{D}_n$ to denote the class of all n-dimensional cdf's on the non-negative orthant of $\mathbb{R}^n$, whereas the symbols $F_i, i = 1, \ldots, n$ are used to denote the marginal cdf's of $F \in \mathcal{D}_n$. The Fréchet class $\mathcal{R}_n$ is defined as follows:

$$\mathcal{R}_n = \{F \in \mathcal{D}_n \mid F_i = F_{X_i},\, i = 1, \ldots, n\}. \tag{58}$$

It is the class of all n-dimensional distribution functions F with marginals F_i equal to the observed risk-neutral distributions F_{X_i} of the r.v.'s X_i.

Because the stop-loss premium $\mathbb{E}\left[(X_i - K)_+\right]$ can be expressed as follows,

$$\mathbb{E}\left[(X_i - K)_+\right] = \int_K^{+\infty} (1 - F_{X_i}(x))\, \mathrm{d}x.$$

any cdf F_i is unambiguously determined by its call or its put option curve. We can define $\mathcal{R}_n$ also as follows:

$$\mathcal{R}_n = \left\{F \in \mathcal{D}_n \mid e^{-rT}\mathbb{E}_{F_i}\left[(X_i - K)_+\right] = C_i[K] \text{ for all } K \text{ and } i = 1, \ldots, n\right\},$$

or

$$\mathcal{R}_n = \left\{F \in \mathcal{D}_n \mid e^{-rT}\mathbb{E}_{F_i}\left[(K - X_i)_+\right] = P_i[K] \text{ for all } K \text{ and } i = 1, \ldots, n\right\},$$

where the subscript denotes the cdf which has to be used to determine the expectation. For example, the notation $\mathbb{E}_{F_i}\left[(X_i - K)_+\right]$ is the stop-loss premium of X_i with retention K, where the cdf of X_i is given by F_i.

This means that $\mathcal{R}_n$ is the class of all n-dimensional cdf's F which produce the observed call and put option curves on the different stocks. In other words, $\mathcal{R}_n$ is the class of all feasible multivariate risk-neutral distributions for $\underline{X}$, given that the only information that we have about $F_{\underline{X}}$ are its marginal distributions F_{X_i}. Obviously, the cdf of $(F_{X_1}^{-1}(U), F_{X_2}^{-1}(U), \ldots, F_{X_n}^{-1}(U))$ is an element of $\mathcal{R}_n$. Knowing that $F_{\underline{X}} \in \mathcal{R}_n$ does not allow us to determine the index option prices $P[K]$ or $C[K]$, however it allows us to determine the comonotonic index option prices $C^c[K]$ and $P^c[K]$.

Theorem 5 (The least upper bound for the index option price) *For any $K \geq 0$ it holds that*

$$\max_{F \in \mathcal{R}_n} \mathrm{e}^{-rT}\mathbb{E}_F\left[(S - K)_+\right] = C^c[K] \tag{59}$$

and

$$\max_{F \in \mathcal{R}_n} \mathrm{e}^{-rT}\mathbb{E}_F\left[(K - S)_+\right] = P^c[K] \tag{60}$$

Moreover, in both cases the maximum is obtained for the cdf of $(F_{X_1}^{-1}(U), F_{X_2}^{-1}(U), \ldots, F_{X_n}^{-1}(U))$.

Proof. For any multivariate distribution F belonging to $\mathcal{R}_n$, we find from the convex order relation (15) that

$$\mathbb{E}_F\left[(S-K)_+\right] \leq \mathbb{E}\left[(S^c-K)_+\right],$$

with $S^c \equiv F_{X_1}^{-1}(U) + \ldots + F_{X_n}^{-1}(U)$. Hence,

$$\sup_{F\in\mathcal{R}_n} \mathbb{E}_F\left[(S-K)_+\right] \leq \mathbb{E}[(S^c-K)_+].$$

On the other hand, the multivariate distribution of $\underline{X}^c \equiv \left(F_{X_1}^{-1}(U), \ldots, F_{X_n}^{-1}(U)\right)$ is an element of $\mathcal{R}_n$. This implies that

$$\mathbb{E}[(S^c-K)_+] \leq \sup_{F\in\mathcal{R}_n} \mathbb{E}_F[(S-K)_+].$$

Combining these observations leads to the call option result in (59). The put option case is proven in a similar way. ■

Theorem 5 states that both upper bounds derived in Theorem 1 can be interpreted as least upper bounds in the sense that they correspond to the largest possible expected discounted pay-off of the corresponding index option, given the risk-neutral distributions of the underlying stocks. Somewhat loosely speaking, $C^c[K]$ is the lowest upper bound for the index call option price $C[K]$ in the class of all models which are consistent with the observed stock option prices. A similar remark holds for the upper bound $P^c[K]$ for the index put option price $P[K]$. The upper bound $C^c[K]$, resp. $P^c[K]$, coincides with the index option prices $C[K]$, resp. $P[K]$, in case the risk-neutral multivariate distribution of the price vector $\underline{X}$ is comonotonic. Notice that in a comonotonic market, all stocks move perfectly together and there is no diversification possible. The question whether it is always possible or not to construct such an arbitrage-free comonotonic market is considered in Hobson et al. (2005) and in Dhaene and Kukush (2010).

4.5 Computational aspects

4.5.1 Numerical evaluation of the upper bounds

In order to be able to calculate the upper bounds $C^c[K]$ and $P^c[K]$ for $K \in \left(F_{S^c}^{-1+}(0), F_{S^c}^{-1}(1)\right)$, one has to determine the probability $F_{S^c}(K)$ and the coefficient α_K.

The coefficient α_K was implicitely defined as any element in $[0,1]$ that satisfies (45). Taking into account the definition (10) of the α-inverse, expression (45) leads to

$$\alpha_K = \begin{cases} \dfrac{F_{S^c}^{-1+}(F_{S^c}(K)) - K}{F_{S^c}^{-1+}(F_{S^c}(K)) - F_{S^c}^{-1}(F_{S^c}(K))}, & \text{if } F_{S^c}^{-1+}(F_{S^c}(K)) \neq F_{S^c}^{-1}(F_{S^c}(K)), \\ 1, & \text{otherwise.} \end{cases} \tag{61}$$

The coefficient α_K follows from this expression, provided we know $F_{S^c}(K)$, $F_{S^c}^{-1}(F_{S^c}(K))$ and $F_{S^c}^{-1+}(F_{S^c}(K))$.

Concerning $F_{S^c}(K)$, notice that

$$F_{S^c}(K) = \sup \{p \in [0,1] \mid F_{S^c}(K) \geq p\}.$$

Using (8) and taking into account the additivity property of quantiles of a comonotonic sum, this relation can be transformed into

$$F_{S^c}(K) = \sup \left\{ p \in [0,1] \mid \sum_{i=1}^{n} w_i F_{X_i}^{-1}(p) \leq K \right\}. \tag{62}$$

Hence $F_{S^c}(K)$ can be determined from the inverse marginal distribution functions $F_{X_i}^{-1}$.

From (17), we find that $F_{S^c}^{-1}(F_{S^c}(K))$ and $F_{S^c}^{-1+}(F_{S^c}(K))$ are given by

$$F_{S^c}^{-1}(F_{S^c}(K)) = \sum_{i=1}^{n} w_i F_{X_i}^{-1}(F_{S^c}(K))$$

and

$$F_{S^c}^{-1+}(F_{S^c}(K)) = \sum_{i=1}^{n} w_i F_{X_i}^{-1+}(F_{S^c}(K)),$$

respectively.

4.5.2 The upper bounds in terms of the inverses $F_{X_i}^{-1}$

The upper bounds (30) and (31) can also be written in terms of the inverses $F_{X_i}^{-1}$, as is shown in the following corollary.

Corollary 1 *For any* $K \in \left(F_{S^c}^{-1+}(0), F_{S^c}^{-1}(1)\right)$ *one has that*

$$\begin{aligned} C^c[K] = &\sum_{i=1}^{n} w_i C_i \left[F_{X_i}^{-1}(F_{S^c}(K))\right] \\ &- e^{-rT}\left(K - F_{S^c}^{-1}(F_{S^c}(K))\right)(1 - F_{S^c}(K)) \end{aligned} \tag{63}$$

and

$$\begin{aligned} P^c[K] = &\sum_{i=1}^{n} w_i P_i \left[F_{X_i}^{-1}(F_{S^c}(K))\right] \\ &+ e^{-rT}\left(K - F_{S^c}^{-1}(F_{S^c}(K))\right) F_{S^c}(K). \end{aligned} \tag{64}$$

Proof. From Dhaene et al. (2000), we find that

$$\begin{aligned} \mathbb{E}\left[(S^c - K)\right] = &\sum_{i=1}^{n} w_i \mathbb{E}\left[\left(X_i - F_{X_i}^{-1}(F_{S^c}(K))\right)_+\right] \\ &- (1 - F_{S^c}(K))\left(K - F_{S^c}^{-1}(F_{S^c}(K))\right). \end{aligned}$$

Combining this expression with (4) proves (63).
Using the put-call parities (21) and (38), expression (63) can be transformed into expression (64). ■

By definition of the α-inverse, it holds that

$$F_{S^c}^{-1}\left(F_{S^c}(K)\right) \leq K = F_{S^c}^{-1(\alpha_K)}\left(F_{S^c}(K)\right), \tag{65}$$

which implies that the second term in the right hand side of (63) is non-negative. Hence, we find from (30) that $\sum_{i=1}^{n} w_i C_i\left[F_{X_i}^{-1}\left(F_{S^c}(K)\right)\right]$ is also an upper bound for the index option price $C[K]$, although it is not necessarily the optimal one in the sense that the time - 0 price of this portfolio of stock options $C_i[F_{X_i}^{-1}(F_{S^c}(K))]$ may not be the the price of the cheapest super-replicating strategy for the index call option $C\left[K\right]$.

Let $K \in \left(F_{S^c}^{-1+}(0), F_{S^c}^{-1}(1)\right)$ and let us assume that all marginal cdf's F_{X_i} are strictly increasing on $\left(F_{X_i}^{-1+}(0), F_{X_i}^{-1}(1)\right)$. This assumption implies that F_{S^c} is strictly increasing on $\left(F_{S^c}^{-1+}(0), F_{S^c}^{-1}(1)\right)$. In this case, any α-inverse $F_{X_i}^{-1(\alpha)}\left(F_{S^c}(K)\right)$ and $F_{S^c}^{-1(\alpha)}\left(F_{S^c}(K)\right)$ coincides with the usual inverse $F_{X_i}^{-1}\left(F_{S^c}(K)\right)$ and $F_{S^c}^{-1(\alpha)}\left(F_{S^c}(K)\right)$, respectively. Furthermore, the comonotonic index option prices (63) and (64) reduce to

$$C^c\left[K\right] = \sum_{i=1}^{n} w_i C_i\left[F_{X_i}^{-1}\left(F_{S^c}(K)\right)\right] \tag{66}$$

and

$$P^c\left[K\right] = \sum_{i=1}^{n} w_i P_i\left[F_{X_i}^{-1}\left(F_{S^c}(K)\right)\right] \tag{67}$$

respectively. From (65) it follows that $F_{S^c}^{-1}\left(F_{S^c}(K)\right) = K$ in this case. Taking into account the additivity property of quantiles of a comonotonic sum, the value $F_{S^c}(K)$ can be obtained from

$$\sum_{i=1}^{n} w_i F_{X_i}^{-1}\left(F_{S^c}(K)\right) = K. \tag{68}$$

If we now additionally assume that at least one cdf F_{X_i} is continuous on $\mathbb{R}$, then one can prove that also F_{S^c} is continuous on $\mathbb{R}$ and $F_{S^c}(K)$ is the unique solution of (68).
A particular situation where the assumptions about the strictly increasingness and the continuity of the marginals F_{X_i} are met and hence, where the expressions (66) and (67) hold is the Black & Scholes model.

5 The finite market case

5.1 Traded options and approximations

In the preceding section, we assumed that the prices $C_i\left[K\right]$ and $P_i\left[K\right]$ of the stock options with maturity T are known for any strike $K \geq 0$. In this section we will investigate the more realistic situation where only finitely many strikes are traded per stock. More specifically, we will assume that for each stock i, only the strikes $K_{i,j}$, $j = 0, 1, \dots, m_i$, are traded and hence, only the prices $C_i[K_{i,j}]$ and $P_i[K_{i,j}]$, $j = 0, 1, \dots, m_i$, are observed. We call the situation where only a finite number of option prices is observed the *finite market case*. As before, we assume that the traded option prices can be expressed as

$$C_i[K_{i,j}] = e^{-rT}\mathbb{E}\left[(X_i - K_{i,j})_+\right], \qquad i = 1, \dots, n;\ j = 0, 1, \dots, m_i, \tag{69}$$

$$P_i[K_{i,j}] = e^{-rT}\mathbb{E}\left[(K_{i,j} - X_i)_+\right], \qquad i = 1, \dots, n;\ j = 0, 1, \dots, m_i, \tag{70}$$

where for each i, the value of stock i at time T is denoted by X_i and the expectations are taken with respect to the distributions F_{X_i} of the stock prices X_i under the risk-neutral measure $\mathbb{Q}$. The only information that we have about these risk-neutral distributions is contained in the observed option prices. Notice that we assume that the sets of traded strikes for the call and put options are identical. This assumption will be relaxed in Section 5.5.

For each stock i, we denote the 'maximal value' of the stock price X_i at time T by K_{i,m_i+1}:

$$F_{X_i}^{-1}(1) := K_{i,m_i+1}. \tag{71}$$

Any value K_{i,m_i+1} may be finite or infinite. In the sequel, we will take a practical approach and assume that all K_{i,m_i+1} are known and have a finite value, which is sufficiently large. Loosely speaking, K_{i,m_i+1} is the maximal possible value for stock i at time T. Appropriate choices for the K_{i,m_i+1} are discussed in Section 5.5.

We assume that the chain of traded strikes is such that

$$0 = K_{i,0} < K_{i,1} < K_{i,2} < \cdots < K_{i,m_i} < K_{i,m_i+1} = F_{X_i}^{-1}(1) < \infty. \tag{72}$$

In particular we assume that for each stock i, the smallest traded strike $K_{i,0}$ is equal to 0.

From (69) and (70) we find that the zero-strike stock option prices are given by

$$C_i[0] = e^{-rT}\mathbb{E}[X_i] \quad \text{and} \quad P_i[0] = 0. \tag{73}$$

Furthermore, from (69) and (70) it follows that the prices of the stock options with strike K_{i,m_i+1} are given by

$$P_i[K_{i,m_i+1}] = e^{-rT}\left(K_{i,m_i+1} - \mathbb{E}[X_i]\right) \quad \text{and} \quad C_i[K_{i,m_i+1}] = 0. \tag{74}$$

Obviously, the put options with strike 0 and the call options with strike K_{i,m_i+1} are not traded. In practice, also the call options with strike 0 and the put options with strike K_{i,m_i+1} are not traded directly. However, these options can be constructed artificially by a combination of traded instruments. For more details we refer to Section 5.5.

An example of observed option curves corresponding to a particular stock at a particular date is given in Figure 1, where the NYSE midquote closing prices for puts and calls on Walt Disney Company are shown. Time 0 is January 23, 2012, whereas the expiration date T is February 17, 2012. The numerical values of these option prices are listed in Table 1.

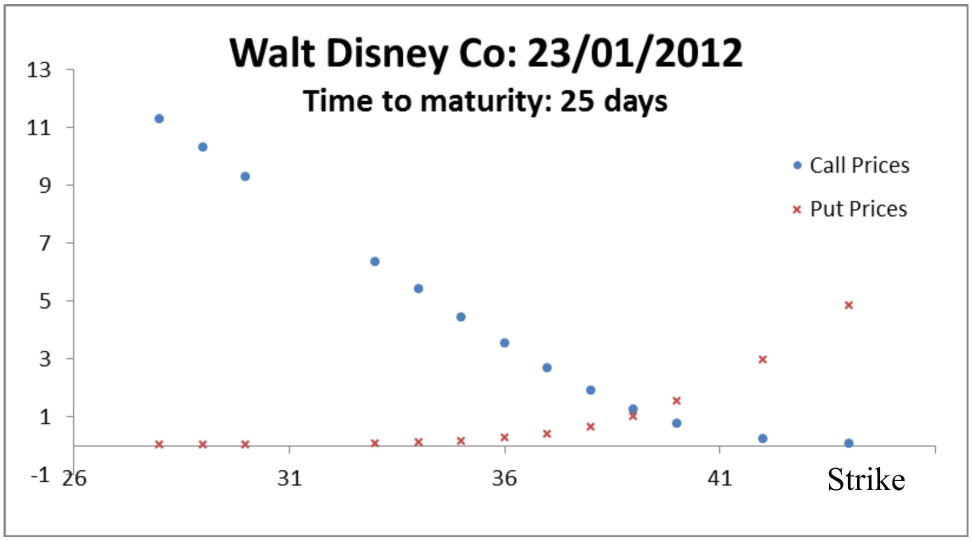

Figure 1: Option curves Walt Disney Co., January 23, 2012.

Strike	Call Price	Put Price
28	11.3	0.015
29	10.3	0.025
30	9.3	0.035
33	6.35	0.08
34	5.425	0.11
35	4.45	0.17
36	3.525	0.265
37	2.68	0.41
38	1.915	0.65
39	1.265	1.01
40	0.775	1.52
42	0.23	2.955
44	0.06	4.825

Table 1: Option prices Walt Disney Co., January 23, 2012.

For each i, we first define the convex functions C_i and P_i by

$$C_i[K] = e^{-rT}\mathbb{E}\left[(X_i - K)_+\right], \qquad K \geq 0 \tag{75}$$

and

$$P_i[K] = e^{-rT}\mathbb{E}\left[(K - X_i)_+\right], \qquad K \geq 0. \tag{76}$$

Notice that in the infinite market case, the value of these functions is known for all K, or equivalently, the risk-neutral distribution F_{X_i} is completely specified for any stock i. On the other hand, in the finite market case, the functions $C_i[K]$ and $P_i[K]$ are known only for the values $K_{i,j}$, $j = 0, 1, \ldots, m_i + 1$, implying that the risk-neutral distributions F_{X_i} are not completely specified.

In Figure 2, the dashed curve corresponds to a possible shape of the curve C_i, of which only the values $C_i[K_{i,j}]$, $j = 0, 1, \ldots, m_i + 1$, are explicitely known.

Let S be the weighted sum of the stock prices X_i at time T, as defined earlier. Suppose that the index call and put options with strike K and respective pay-offs $(S - K)_+$ and $(K - S)_+$ at time T are traded in the market. Their prices are denoted by $C[K]$ and $P[K]$. As before, we assume that these prices can be expressed as

$$C[K] = e^{-rT}\mathbb{E}\left[(S - K)_+\right] \tag{77}$$

and

$$P[K] = e^{-rT}\mathbb{E}\left[(K - S)_+\right], \tag{78}$$

where the expectations are taken with respect to the distribution F_S of S under the $\mathbb{Q}$-measure.

It is our goal to find upper bounds for the index option prices $C[K]$ and $P[K]$ which can be expressed in terms of the available stock option prices $C_i[K_{i,j}]$ and $P_i[K_{i,j}]$, $j = 0, 1, \ldots, m_i + 1$. We will show that the solution to this problem follows in a rather straightforward way from the results derived for the infinite market case.

From Theorem 2 we find the following upper bounds for the index option prices :

$$C[K] \leq \sum_{i=1}^{n} w_i C_i[K_i^*] \text{ and } P[K] \leq \sum_{i=1}^{n} w_i P_i[K_i^*],$$

with the K_i^* defined in (42). In the finite market case, it is in general not possible to determine these upper bounds numerically, because the distribution function of S^c is not completely specified. In order to solve this problem, in a first step we construct approximations $\overline{C}_i$ and $\overline{P}_i$ for the functions C_i and P_i respectively, which are fully specified. In particular, we define $\overline{C}_i$ and $\overline{P}_i$ as the piecewise linear functions connecting the observed points $(K_{i,j}, C_i[K_{i,j}])$ and $(K_{i,j}, P_i[K_{i,j}])$, $j = 0, 1, \ldots, m_i + 1$, respectively. Hence, $\overline{C}_i$ and $\overline{P}_i$ are piecewise linear functions, changing their slope only in the observed strikes $K_{i,j}$ and such that

$$\overline{C}_i[K_{i,j}] = C_i[K_{i,j}], \qquad j = 0, 1, \ldots, m_i + 1, \tag{79}$$

$$\overline{P}_i[K_{i,j}] = P_i[K_{i,j}], \qquad j = 0, 1, \ldots, m_i + 1. \tag{80}$$

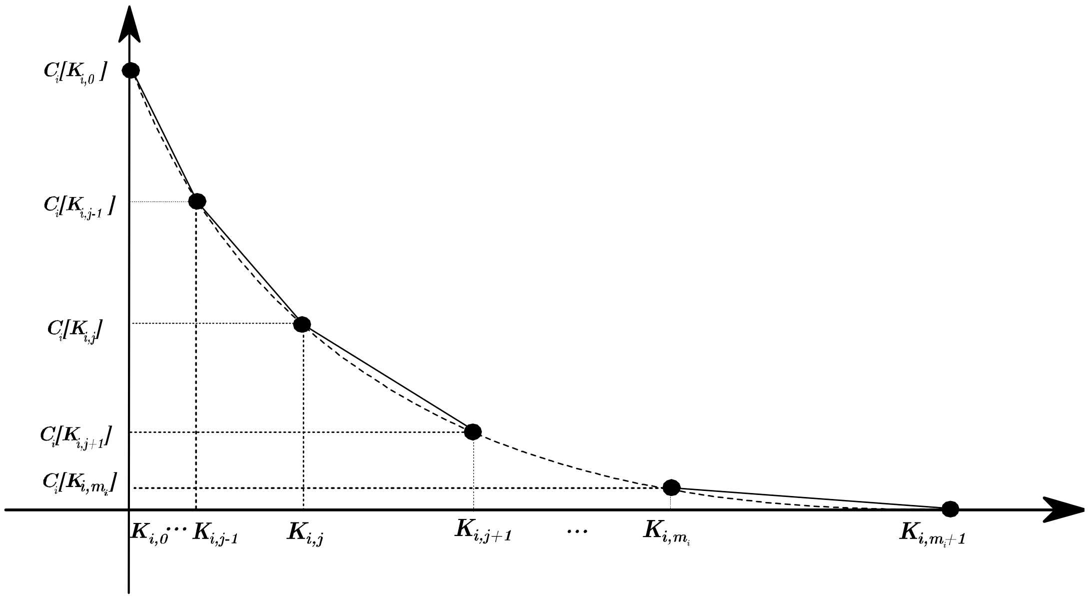

Figure 2: The option curves $C_i[K]$ (dashed curve) and $\overline{C}_i[K]$ (solid curve).

From (75) and (76) it follows that

$$C_i[K] = C_i[0] - e^{-rT}K \text{ and } P_i[K] = 0, \qquad \text{if } K \leq 0,$$

whereas

$$C_i[K] = 0 \text{ and } P_i[K] = e^{-rT}K - C_i[0], \qquad \text{if } K \geq K_{i,m_i+1}.$$

Therefore, we define $\overline{C}_i$ and $\overline{P}_i$ as follows in the region outside $(0, K_{i,m_i+1})$:

$$\overline{C}_i[K] = C_i[K] \qquad \text{if } K \notin (0, K_{i,m_i+1}), \tag{81}$$

$$\overline{P}_i[K] = P_i[K] \qquad \text{if } K \notin (0, K_{i,m_i+1}) \tag{82}$$

In Figure 2, the dashed curve corresponds to the (unknown) option curve $C_i[K]$, whereas the solid curve corresponds to the piecewise linear approximation $\overline{C}_i[K]$.

The results for the infinite market case derived in the previous section will be applied to the piecewise linear curves $\overline{C}_i$ and $\overline{P}_i$. This will lead to upper bounds in terms of stock options for appropriately defined strikes K_i^*. At first sight, one may end up with the case where the upper bounds contain stock options with strikes K_i^* that are not traded in the market. However, we will show that for any 'unreachable' strike, the corresponding call or put stock option price can be expressed in terms of a convex combination of its neighbouring observed stock option prices.

In the following lemma, we consider the piecewise linear approximation $\overline{C}_i$ for the call option curve C_i.

Lemma 1 (Piecewise linear approximation for the call option curve) *The piecewise linear approximation $\overline{C}_i$ for the stock option curve C_i is given by*

$$\overline{C}_i\left[K\right] = \frac{C_i\left[K_{i,j+1}\right] - C_i\left[K_{i,j}\right]}{K_{i,j+1} - K_{i,j}}\left(K - K_{i,j}\right) + C_i\left[K_{i,j}\right], \tag{83}$$

in case $K_{i,j} \leq K < K_{i,j+1}$, $j = 0, 1, \ldots, m_i$.
For $K \leq 0$, it is given by

$$\overline{C}_i\left[K\right] = C_i\left[K\right] = C_i\left[0\right] - \mathrm{e}^{-rT}K, \tag{84}$$

while for $K \geq K_{i,m_i+1}$, one has that

$$\overline{C}_i\left[K\right] = C_i\left[K\right] = 0. \tag{85}$$

The function $\overline{C}_i$ is convex and decreasing. Furthermore,

$$\overline{C}_i[K] \geq C_i[K] \text{ for all } K.$$

Proof. Expression (83) follows from the fact that the line that connects the observed points $(K_{i,j}; C_i\left[K_{i,j}\right])$ and $(K_{i,j+1}; C_i\left[K_{i,j+1}\right])$ is given by (83). The expressions (84) and (85) hold by definition of $\overline{C}_i$. The convexity and decreasingness of $\overline{C}_i$ follows from the corresponding properties of C_i. ■

Next, we consider the piecewise linear approximation $\overline{P}_i$ for the put option curve P_i.

Lemma 2 (Piecewise linear approximation for the put option curve) *The piecewise linear approximation $\overline{P}_i$ for the option curve P_i is given by*

$$\overline{P}_i\left[K\right] = \frac{P_i\left[K_{i,j+1}\right] - P_i\left[K_{i,j}\right]}{K_{i,j+1} - K_{i,j}}\left(K - K_{i,j}\right) + P_i\left[K_{i,j}\right],$$

in case $K_{i,j} \leq K < K_{i,j+1}$, $j = 0, 1, \ldots, m_i$.
For $K \leq 0$, it is given by

$$\overline{P}_i\left[K\right] = P_i\left[K\right] = 0,$$

while for $K \geq K_{i,m_i+1}$, one has that

$$\overline{P}_i\left[K\right] = P_i\left[K\right] = \mathrm{e}^{-rT}K - C_i\left[0\right].$$

The function $\overline{P}_i$ is convex and increasing. Furthermore,

$$\overline{P}_i[K] \geq P_i[K] \text{ for all } K.$$

Proof. The proof is similar to the proof of Lemma 1. ■

From the previous lemma's one can prove that the following put-call parity holds for the approximated stock option curves:

$$\overline{C}_i\left[K\right] + e^{-rT}K = \overline{P}_i\left[K\right] + e^{-rT}\mathbb{E}\left[X_i\right]. \tag{86}$$

In the infinite market case, we were able to obtain the risk-neutral cdf F_{X_i} of X_i from the observed stock option curve $C_i\left[K\right]$ via expression (26) or from the observed put option curve $P_i\left[K\right]$ via expression (27). In the finite market case, we are not able to determine F_{X_i}. In a first step, we proposed to approximate the partially known call and put option curves $C_i\left[K\right]$ and $P_i\left[K\right]$ by the completely specified piecewise linear functions $\overline{C}_i[K]$ and $\overline{P}_i[K]$, respectively. In a second step, we will determine the distribution functions $\overline{F}_{X_i}$ such that the approximated option prices $\overline{C}_i[K]$ and $\overline{P}_i[K]$ can be expressed as expected values of the respective discounted pay-offs, where the expectations are taken with respect to that distribution. In the following lemma, we consider the call option case.

Lemma 3 (The cdf $\overline{F}_{X_i}$ of X_i corresponding to $\overline{C}_i$) *Let $\overline{F}_{X_i}$ be the cdf of X_i determined such that*

$$e^{-rT}\mathbb{E}_{\overline{F}_{X_i}}\left[(X_i-K)_+\right]=\overline{C}_i\left[K\right], \qquad \text{for all } K. \tag{87}$$

Then we have that

$$\overline{F}_{X_i}(x)=\begin{cases} 0 & \text{if } x<0, \\ 1+e^{rT}\dfrac{C_i\left[K_{i,j+1}\right]-C_i\left[K_{i,j}\right]}{K_{i,j+1}-K_{i,j}} & \text{if } K_{i,j}\leq x<K_{i,j+1}, \quad j=0,1,\ldots,m_i, \\ 1 & \text{if } x\geq K_{i,m_i+1}. \end{cases} \tag{88}$$

Proof. For the particular situation at hand, the expression (26) translates into

$$\overline{F}_{X_i}(x)=1+e^{rT}\overline{C}_i'[x+].$$

The proof of (88) follows immediately from applying this expression to the function $\overline{C}_i[K]$ defined in Lemma 1. ■

Let us now consider the put option case.

Lemma 4 (The cdf $\overline{F}_{X_i}$ of X_i corresponding to $\overline{P}_i$) *Let $\overline{F}_{X_i}$ be the cdf of X_i determined such that*

$$e^{-rT}\mathbb{E}_{\overline{F}_{X_i}}\left[(K-X_i)_+\right]=\overline{P}_i\left[K\right] \qquad \text{for all } K. \tag{89}$$

Then we have that

$$\overline{F}_{X_i}(x)=\begin{cases} 0 & \text{if } x<0, \\ e^{rT}\dfrac{P_i\left[K_{i,j+1}\right]-P_i\left[K_{i,j}\right]}{K_{i,j+1}-K_{i,j}} & \text{if } K_{i,j}\leq x<K_{i,j+1}, \quad j=0,1,\ldots,m_i, \\ 1 & \text{if } x\geq K_{i,m_i+1}. \end{cases} \tag{90}$$

Proof. Translating expression (27) to the situation at hand, we find that

$$\overline{F}_{X_i}(x)=e^{rT}\overline{P}_i'[x+].$$

The proof of (90) follows then from applying this expression to the function $\overline{P}_i[K]$ defined in Lemma 2. ∎

Important to notice is that the put-call parity (86) allows us to prove that the underlying distribution $\overline{F}_{X_i}$ derived from the call option curve $\overline{C}_i[K]$ is equal to the distribution function that emerged from the put option curve $\overline{P}_i[K]$.

From Lemma 1 we find the following ordering relations between the distributions F_{X_i} and $\overline{F}_{X_i}$:

$$X_i \stackrel{\mathrm{d}}{=} F_{X_i}^{-1}(U) \leq_{\mathrm{cx}} \overline{F}_{X_i}^{-1}(U), \tag{91}$$

where as usual, U is a r.v. which is uniformly distributed over the unit interval.

For any stock i, we have that $\overline{F}_{X_i}$ is a discrete distribution function, with possible outcomes given by the traded strikes $K_{i,j}$. For any $x \in [K_{i,j}, K_{i,j+1})$, $j = 0, 1, \ldots, m_i$, one has that $0 \leq F_{X_i}(x) < 1$. The first strictly positive jump upwards of $\overline{F}_{X_i}(x)$ does not necessarily occurs at 0, but the last strictly positive jump upwards of $\overline{F}_{X_i}(x)$ always occurs at K_{i,m_i+1}. Therefore, we have to determine $\overline{F}_{X_i}^{-1+}(0)$ and $\overline{F}_{X_i}^{-1}(1)$ as follows:

$$\overline{F}_{X_i}^{-1+}(0) = \min_{j\in\{0,1,\ldots,m_i\}} \left\{K_{i,j} \mid \overline{F}_{X_i}(K_{i,j}) > 0\right\} \text{ and } \overline{F}_{X_i}^{-1}(1) = K_{i,m_i+1}. \tag{92}$$

A possible shape of the risk-neutral cdf $\overline{F}_{X_i}$ of stock i is shown in Figure 3. In this particular case, we have that $\overline{F}_{X_i}^{-1+}(0) = K_{i,2}$. Figure 4 shows the corresponding option curves $\overline{C}_i[K]$ and $\overline{P}_i K]$.

Hereafter, we will always silently assume that

$$\overline{F}_{X_i}(K_{i,m_i}) > 0, \qquad i = 1, 2, \ldots, n. \tag{93}$$

This assumption means that no marginal cdf $\overline{F}_{X_i}$ has a one-point distribution. Notice that this assumption can always be satisfied by chosing the maximal values K_{i,m_i+1}, $i = 1, 2, \ldots, n$, sufficiently large.

Taking into account Lemma 3 and expression (69) for the call option prices, we find the following relation between the cdf's $\overline{F}_{X_i}$ and F_{X_i}:

$$\overline{F}_{X_i}(K_{i,j}) = \frac{1}{K_{i,j+1} - K_{i,j}} \int_{K_{i,j}}^{K_{i,j+1}} F_{X_i}(x)\,\mathrm{d}x, \qquad j = 0, 1, \ldots, m_i, \tag{94}$$

where we used the following representation for the stop-loss premiums of X_i:

$$\mathbb{E}\left[(X_i - K)_+\right] = \int_K^{+\infty} (1 - F_{X_i}(x))\,\mathrm{d}x.$$

From (92) we find that for any $j = 0, 1, \ldots, m_i$, it holds that

$$\overline{F}_{X_i}^{-1+}(0) = K_{i,j} \Longleftrightarrow \overline{F}_{X_i}(K_{i,j-1}) = 0 \text{ and } \overline{F}_{X_i}(K_{i,j}) > 0, \tag{95}$$

where $K_{i,-1}$ is defined by

$$K_{i,-1} = -1. \tag{96}$$

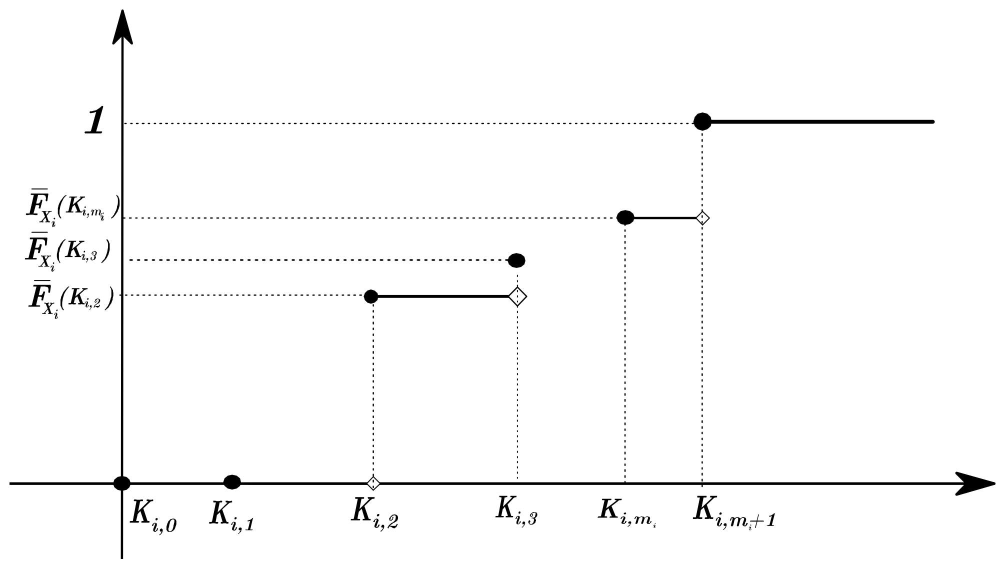

Figure 3: The cdf $\overline{F}_{X_i}$ of X_i.

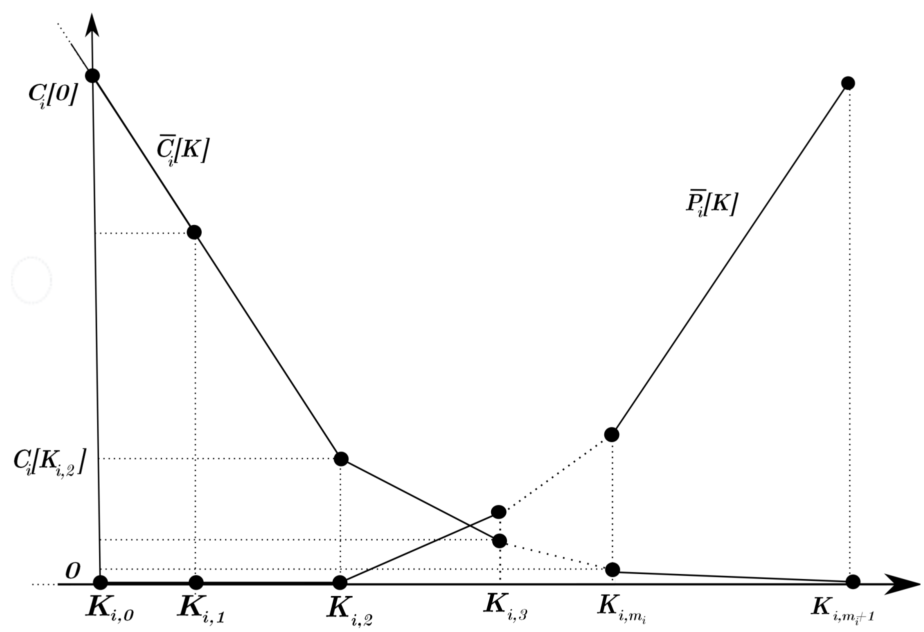

Figure 4: The curves $\overline{C}_i\,[K]$ and $\overline{P}_i\,[K]$ under $\overline{F}_{X_i}$.

Obviously, one has that

$$\overline{F}_{X_i}(K_{i,-1}) = 0. \tag{97}$$

Taking into account (94), the equivalence relations (95) can be rewritten as

$$\overline{F}_{X_i}^{-1+}(0) = K_{i,j} \iff K_{i,j} \leq F_{X_i}^{-1+}(0) < K_{i,j+1}, \qquad j = 0, 1, \ldots, m_i. \tag{98}$$

This means that $\overline{F}_{X_i}^{-1+}(0)$ is equal to $K_{i,j}$ when the 'smallest value' of X_i is contained in the interval $[K_{i,j}, K_{i,j+1})$. In particular, we have that

$$\overline{F}_{X_i}^{-1+}(0) = 0 \iff 0 \leq F_{X_i}^{-1+}(0) < K_{i,1}. \tag{99}$$

Hence, $\overline{F}_{X_i}^{-1+}(0) = 0$ if the 'minimal possible value' of the price X_i of stock i at time T is strictly smaller than strike $K_{i,1}$.

5.2 An upper bound for the index option price

Our goal is to find the best possible upper bound for the prices $C[K]$ and $P[K]$ of the traded index options in terms of the observed stock option prices $C_i[K_{i,j}]$ and $P_i[K_{i,j}]$. This upper bound will be expressed in terms of the comonotonic sum $\overline{S}^c$, which is defined by

$$\overline{S}^c = w_1 \overline{F}_{X_1}^{-1}(U) + w_2 \overline{F}_{X_2}^{-1}(U) + \cdots + w_n \overline{F}_{X_n}^{-1}(U). \tag{100}$$

The extreme outcomes of $\overline{S}^c$ fulfill the following conditions:

$$F_{\overline{S}^c}^{-1+}(0) = \sum_{i=1}^{n} w_i \overline{F}_{X_i}^{-1+}(0) \leq \sum_{i=1}^{n} w_i F_{X_i}^{-1+}(0) = F_{S^c}^{-1+}(0) \leq F_S^{-1+}(0), \tag{101}$$

$$F_S^{-1}(1) \leq F_{S^c}^{-1}(1) = \sum_{i=1}^{n} w_i K_{i,m_i+1} = F_{\overline{S}^c}^{-1}(1). \tag{102}$$

In the following theorem, we derive upper bounds for the index option prices $C[K]$ and $P[K]$ in terms of the distribution function of $\overline{S}^c$.

Theorem 6 (An upper bound for the index option price) *The prices $C[K]$ and $P[K]$ of the traded index options with pay-off $(S-K)_+$ and $(K-S)_+$ at time T are constrained from above as follows:*

$$C[K] \leq e^{-rT} \mathbb{E}\left[\left(\overline{S}^c - K\right)_+\right], \tag{103}$$

$$P[K] \leq e^{-rT} \mathbb{E}\left[\left(K - \overline{S}^c\right)_+\right]. \tag{104}$$

Proof. From (15), (16) and (91) we find that

$$\sum_{i=1}^{n} w_i X_i \leq_{\text{cx}} \sum_{i=1}^{n} w_i F_{X_i}^{-1}(U) \leq_{\text{cx}} \sum_{i=1}^{n} w_i \overline{F}_{X_i}^{-1}(U),$$

or equivalently,

$$S \leq_{\text{cx}} S^c \leq_{\text{cx}} \overline{S}^c. \tag{105}$$

The stated inequalities follow from (6). ∎

The right-hand sides of (103) and (104) correspond to the prices of an index call and put option with strike K in case the stock option curves are piecewise linear and moreover, the dependence structure between the stock prices is the comonotonic one. In the sequel we will use the notations $\overline{C}^c[K]$ and $\overline{P}^c[K]$ for options written on $\overline{S}^c$:

$$\overline{C}^c[K] = e^{-rT}\mathbb{E}\left[\left(\overline{S}^c - K\right)_+\right], \tag{106}$$

$$\overline{P}^c[K] = e^{-rT}\mathbb{E}\left[\left(K - \overline{S}^c\right)_+\right], \tag{107}$$

and call them the *comonotonic index call and put option prices.* Notice that the following put-call parity holds for these comonotonic option prices:

$$\overline{C}^c[K] + e^{-rT}K = \overline{P}^c[K] + e^{-rT}\mathbb{E}[S]. \tag{108}$$

For $K \notin \left(F_{\overline{S}^c}^{-1+}(0), F_{\overline{S}^c}^{-1}(1)\right)$, we know the exact value of the index option prices $C[K]$ and $P[K]$; see (34) and (35). Furthermore, one has that

$$C[K] = \overline{C}^c[K] \qquad \text{if } K \notin \left(F_{\overline{S}^c}^{-1+}(0), F_{\overline{S}^c}^{-1}(1)\right),$$
$$P[K] = \overline{P}^c[K] \qquad \text{if } K \notin \left(F_{\overline{S}^c}^{-1+}(0), F_{\overline{S}^c}^{-1}(1)\right).$$

As the values of $C[K]$ and $P[K]$ are explicitely known when $K \notin \left(F_{\overline{S}^c}^{-1+}(0), F_{\overline{S}^c}^{-1}(1)\right)$, in the sequel we will focus on the case where $K \in \left(F_{\overline{S}^c}^{-1+}(0), F_{\overline{S}^c}^{-1}(1)\right)$ when considering upper bounds for the index option prices. When not explicitely mentioned, we will always suppose that $K \in \left(F_{\overline{S}^c}^{-1+}(0), F_{\overline{S}^c}^{-1}(1)\right)$.

In the following theorem, we show that the upper bounds derived in Theorem 6 can be expressed in terms of stock option prices.

Theorem 7 (Expressions for $\overline{C}^c$ and $\overline{P}^c$) *For any $K \in \left(F_{\overline{S}^c}^{-1+}(0), F_{\overline{S}^c}^{-1}(1)\right)$, the comonotonic option prices $\overline{C}^c[K]$ and $\overline{P}^c[K]$ can be expressed as*

$$\overline{C}^c[K] = \sum_{i=1}^{n} w_i \overline{C}_i[K_i^*], \tag{109}$$

$$\overline{P}^c[K] = \sum_{i=1}^{n} w_i \overline{P}_i[K_i^*], \tag{110}$$

with the K_i^* given by

$$K_i^* = \overline{F}_{X_i}^{-1(\alpha_K)}\left(F_{\overline{S}^c}(K)\right), \qquad i = 1, 2, \ldots, n \tag{111}$$

and where α_K is any element in $[0,1]$ such that

$$\sum_{i=1}^{n} w_i K_i^* = K. \tag{112}$$

Proof. The proof of the stated results is similar to the proof of Theorem 2. ■

From Lemma 1, we know that for each i, the comonotonic option price $\overline{C}_i\left[K_i^*\right]$ can be expressed in terms of at most two observed option prices $C_i\left[K_{i,j}\right]$, $j = 0, 1, \ldots, m_i + 1$. Hence, the upper bound $\overline{C}^c\left[K\right]$ for the index call option price $C\left[K\right]$ is a linear combination of observed stock call option prices. A similar remark holds for the index put option.

Taking into account the additivity property (17) for quantiles of a comonotonic sum, relation (112) can be rewritten as

$$F_{\overline{S}^c}^{-1(\alpha_K)}\left(F_{\overline{S}^c}(K)\right) = K. \tag{113}$$

Hereafter, we explain how to determine the upper bounds $\overline{C}^c\left[K\right]$ and $\overline{P}^c\left[K\right]$. Therefore, we first introduce the indices $j_i(K)$ and the sets N_K and $\overline{N}_K$.

Let $K \in \left(F_{\overline{S}^c}^{-1+}(0), F_{\overline{S}^c}^{-1}(1)\right)$, then we have that $F_{\overline{S}^c}(K) \in (0,1)$. For any such K and any stock i, we define $j_i(K) \equiv j_i$ as the unique element contained in the set $\{0, 1, \ldots, m_i + 1\}$ that satisfies

$$\overline{F}_{X_i}(K_{i,j_i-1}) < F_{\overline{S}^c}(K) \leq \overline{F}_{X_i}(K_{i,j_i}). \tag{114}$$

Further, we define the set N_K as follows:

$$N_K = \left\{ i \in \{1, 2, \ldots, n\} \mid \overline{F}_{X_i}(K_{i,j_i-1}) < F_{\overline{S}^c}(K) < \overline{F}_{X_i}(K_{i,j_i}) \right\}. \tag{115}$$

Its complement $\overline{N}_K$ is the set given by

$$\overline{N}_K = \left\{ i \in \{1, 2, \ldots, n\} \mid F_{\overline{S}^c}(K) = \overline{F}_{X_i}(K_{i,j_i}) \right\}. \tag{116}$$

Notice that $i \in \overline{N}_K$ implies that $j_i \in \{0, 1, \ldots, m_i\}$.

In Figures 5 and 6, we illustrate how to determine the indices j_i. In Figure 5 we consider the case where $\overline{F}_{X_i}(K_{i,j-1}) < F_{\overline{S}^c}(K) < \overline{F}_{X_i}(K_{i,j})$, hence $i \in N_K$. In Figure 6, we have that $F_{\overline{S}^c}(K) = \overline{F}_{X_i}(K_{i,j})$, which implies that $i \in \overline{N}_K$.

In the following theorem, we prove that the comonotonic option prices $\overline{C}^c\left[K\right]$ and $\overline{P}^c\left[K\right]$ for the index call and put options can be expressed in terms of traded stock option prices.

Theorem 8 ($\overline{C}^c$ and $\overline{P}^c$ are l.c.'s of stock option prices) *For any $K \in \left(F_{\overline{S}^c}^{-1+}(0), F_{\overline{S}^c}^{-1}(1)\right)$, the comonotonic option prices $\overline{C}^c\left[K\right]$ and $\overline{P}^c\left[K\right]$ can be expressed as*

$$\overline{C}^c\left[K\right] = \sum_{i \in N_K} w_i C_i\left[K_{i,j_i}\right] + \sum_{i \in \overline{N}_K} w_i \left(\alpha_K C_i\left[K_{i,j_i}\right] + (1-\alpha_K) C_i\left[K_{i,j_i+1}\right]\right), \tag{117}$$

$$\overline{P}^c\left[K\right] = \sum_{i \in N_K} w_i P_i\left[K_{i,j_i}\right] + \sum_{i \in \overline{N}_K} w_i \left(\alpha_K P_i\left[K_{i,j_i}\right] + (1-\alpha_K) P_i\left[K_{i,j_i+1}\right]\right), \tag{118}$$

where α_K is any element in $[0,1]$ such that (113) holds.

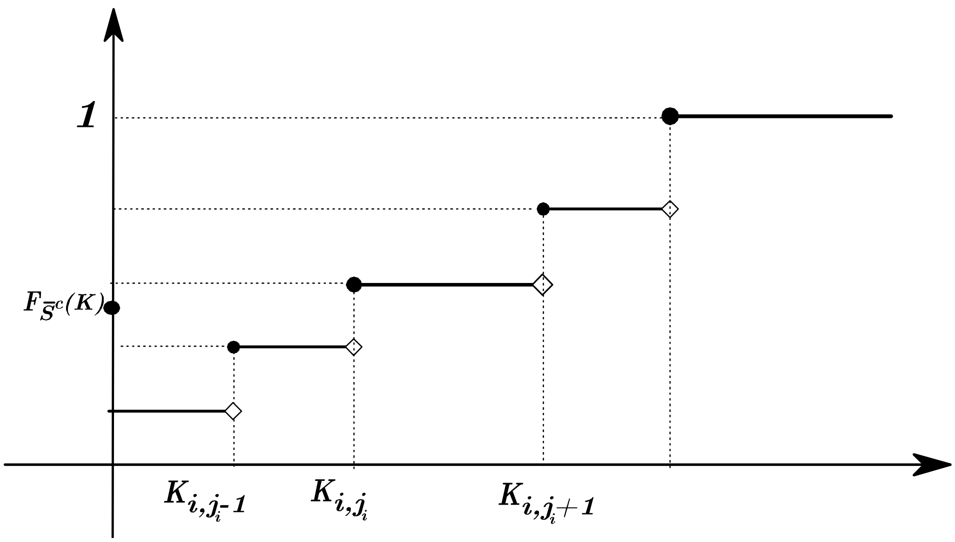

Figure 5: The cdf $\overline{F}_{X_i}(x)$ in case $i \in N_K$.

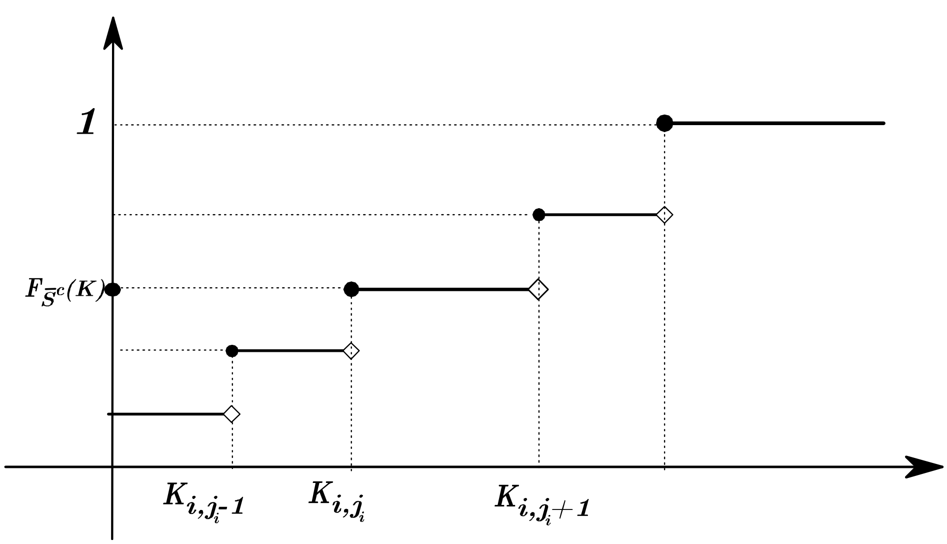

Figure 6: The cdf $\overline{F}_{X_i}(x)$ in case $i \in \overline{N}_K$.

Proof. Let $K \in \left(F_{\overline{S}^c}^{-1+}(0), F_{\overline{S}^c}^{-1}(1)\right)$. From Lemma 3 it follows that for any $\alpha \in [0,1]$, the α-quantile $\overline{F}_{X_i}^{-1(\alpha)}(p)$, $0 < p < 1$, is given by

$$\overline{F}_{X_i}^{-1(\alpha)}(p) = \begin{cases} K_{i,j} & \text{if } \overline{F}_{X_i}(K_{i,j-1}) < p < \overline{F}_{X_i}(K_{i,j}), \\ & \quad j = 0, 1, \ldots, m_i + 1, \\ \alpha K_{i,j} + (1-\alpha) K_{i,j+1} & \text{if } p = \overline{F}_{X_i}(K_{i,j}),\ j = 0 \ldots, m_i. \end{cases} \tag{119}$$

Taking into account the definitions of the indices j_i and the sets N_K and $\overline{N}_K$ in (114), (115) and (116), we find that

$$\overline{F}_{X_i}^{-1(\alpha)}(F_{\overline{S}^c}(K)) = \begin{cases} K_{i,j_i} & \text{if } i \in N_K \\ \alpha K_{i,j_i} + (1-\alpha) K_{i,j_i+1} & \text{if } i \in \overline{N}_K \end{cases} \tag{120}$$

holds for any $\alpha \in [0,1]$.
Combining (120) with Lemma 1 and using the linearity of the function $\overline{C}_i$, we arrive at

$$\overline{C}_i\left[\overline{F}_{X_i}^{-1(\alpha)}(F_{\overline{S}^c}(K))\right] = \begin{cases} \overline{C}_i\left[K_{i,j_i}\right] & \text{if } i \in N_K \\ \overline{C}_i\left[\alpha K_{i,j_i} + (1-\alpha) K_{i,j_i+1}\right] & \text{if } i \in \overline{N}_K \end{cases}$$

$$= \begin{cases} C_i\left[K_{i,j_i}\right] & \text{if } i \in N_K \\ \alpha C_i\left[K_{i,j_i}\right] + (1-\alpha) C_i\left[K_{i,j_i+1}\right] & \text{if } i \in \overline{N}_K. \end{cases} \tag{121}$$

which holds for any α in $[0,1]$. The proof of (117) follows from Theorem 7 and expression (121) for $\alpha = \alpha_K$.
Expression (118) can be proven in a similar way or via the put-call parities (21) for stock option prices and (108) for comonotonic index option prices. ■

In order to calculate the comonotonic option prices (117) and (118) in Theorem 8, we first have to determine $F_{\overline{S}^c}(K)$ and α_K. Knowledge of $F_{\overline{S}^c}(K)$ allows to determine the indices j_i, as well as the sets N_K and $\overline{N}_K$. The numerical valuation of these quantities is considered in Section 5.5.

In the following theorem we prove that each upper bound presented in the previous theorem corresponds to the price of a static super-replicating strategy for the index option under consideration.

Theorem 9 ($\overline{C}^c$ and $\overline{P}^c$ are the prices of static super-replicating strategies) *Let $K \in \left(F_{\overline{S}^c}^{-1+}(0), F_{\overline{S}^c}^{-1}(1)\right)$ and consider the index call and put options with pay-off at time T given by $(S-K)_+$ and $(K-S)_+$, respectively.*

1. *Consider the strategy where at time 0, for any stock $i \in N_K$ one buys w_i calls $C_i\left[K_{i,j_i}\right]$, while for any stock $i \in \overline{N}_K$ one buys $\alpha_K w_i$ calls $C_i\left[K_{i,j_i}\right]$ and $(1-\alpha_K)\, w_i$ calls $C_i\left[K_{i,j_i+1}\right]$. Furthermore, these positions are held until they expire at time T. This static strategy super-replicates the pay-off of the index call option with price $C[K]$. Its price is given by $\overline{C}^c[K]$.*

2. *Consider the pay-off of the strategy where at time* 0*, for any stock* $i \in N_K$ *one buys* w_i *puts* $P_i\left[K_{i,j_i}\right]$*, while for any stock* $i \in \overline{N}_K$ *one buys* $\alpha_K w_i$ *puts* $P_i\left[K_{i,j_i}\right]$ *and* $(1-\alpha_K)\, w_i$ *puts* $P_i\left[K_{i,j_i+1}\right]$*. Furthermore, these positions are held until they expire at time* T*. This static strategy super-replicates the pay-off of the index put option with price* $P\left[K\right]$*. Its price is given by* $\overline{P}^c\left[K\right]$*.*

Proof. The pay-off of the first strategy described in the theorem is given by

$$\sum_{i\in N_K} w_i \left(X_i - K_{i,j_i}\right)_+ + \sum_{i\in \overline{N}_K} w_i \left(\alpha_K \left(X_i - K_{i,j_i}\right)_+ + (1-\alpha_K)\left(X_i - K_{i,j_i+1}\right)_+\right),$$

while from Theorem 8 it follows that its price is given by $\overline{C}^c\left[K\right]$.
Taking into account (112) we find that the pay-off at time T of the index option $C\left[K\right]$ can be expressed as

$$(S-K)_+ = \left(\sum_{i=1}^{n} w_i \left(X_i - K_i^*\right)\right)_+, \tag{122}$$

with the K_i^* defined in (111).
It remains to prove that

$$(S-K)_+ \leq \sum_{i\in N_K} w_i \left(X_i - K_{i,j_i}\right)_+ + \sum_{i\in \overline{N}_K} w_i \left(\alpha_K \left(X_i - K_{i,j_i}\right)_+ + (1-\alpha_K)\left(X_i - K_{i,j_i+1}\right)_+\right). \tag{123}$$

In order to prove this inequality, observe from (120) that

$$K_i^* = \overline{F}_{X_i}^{-1(\alpha_K)}(F_{\overline{S}^c}(K)) = \begin{cases} K_{i,j_i} & \text{if } i \in N_K, \\ \alpha_K K_{i,j_i} + (1-\alpha_K) K_{i,j_i} & \text{if } i \in \overline{N}_K. \end{cases} \tag{124}$$

Taking into account (112) and (124), we find from (122) that

$$\begin{aligned} (S-K)_+ &\leq \sum_{i=1}^{n} w_i \left(X_i - K_i^*\right)_+ \\ &= \sum_{i\in N_K} w_i \left(X_i - K_{i,j_i}\right)_+ + \sum_{i\in \overline{N}_K} w_i \left(X_i - \alpha_K K_{i,j_i} - (1-\alpha_K) K_{i,j_i+1}\right)_+ \\ &\leq \sum_{i\in N_K} w_i \left(X_i - K_{i,j_i}\right)_+ + \sum_{i\in \overline{N}_K} w_i \left(\alpha_K \left(X_i - K_{i,j_i}\right)_+ + (1-\alpha_K)\left(X_i - K_{i,j_i+1}\right)_+\right), \end{aligned}$$

so we have proven that the first strategy in the theorem indeed super-replicates the index call option pay-off.

Let us now consider the put option case. Using the relation

$$(S-K)_+ = (K-S)_+ + S - K,$$

one immediately finds from (123) that

$$(K-S)_+ \leq \sum_{i\in N_K} w_i \left(K_{i,j_i} - X_i\right)_+ + \sum_{i\in \overline{N}_K} w_i \left(\alpha_K \left(K_{i,j_i} - X_i\right)_+ + (1-\alpha_K)\left(K_{i,j_i+1} - X_i\right)_+\right).$$

This inequality proves that the second strategy in the theorem is indeed a super-replicating strategy for the the index put option $P[K]$. From Theorem 8, it follows that the price of this strategy is given by $\overline{P}^c[K]$. ■

From our previous derivations, we can conclude that the following inequalities hold concerning the index option prices:

$$C[K] \leq \sum_{i \in N_K} w_i C_i[K_{i,j_i}] + \sum_{i \in \overline{N}_K} w_i \left(\alpha_K\ C_i[K_{i,j_i}] + (1-\alpha_K) C_i[K_{i,j_i+1}]\right), \tag{125}$$

$$P[K] \leq \sum_{i \in N_K} w_i P_i[K_{i,j_i}] + \sum_{i \in \overline{N}_K} w_i \left(\alpha_K\ P_i[K_{i,j_i}] + (1-\alpha_K) P_i[K_{i,j_i+1}]\right). \tag{126}$$

The right hand side of equation (125) is the price of a static super-replicating strategy for the index call option with pay-off $(S-K)_+$ at time T, whereas the right hand side of equation (126) is the price of a static super-replicating strategy for the index put option with pay-off $(K-S)_+$ at time T. From these observations we can conclude that the upper bound inequalities (125) and (126) remain to hold, without having to make the explicit assumption that the involved option prices are expected discounted pay-offs under some $\mathbb{Q}$-measure. The only assumption that we have made is that the market is free of arbitrage. Remark however that in order to prove the equalities (117) and (118), we have to make the assumption that any option price can be expressed as an expectation of its discounted pay-off.

5.3 The upper bound is the price of the cheapest super-replicating strategy

The upper bounds (125) and (126) for the index option prices $C[K]$ and $P[K]$ are both linear combinations of observed stock options prices. Each bound can be interpreted as the price of a static strategy that super-replicates the pay-off of the corresponding index option; see Theorem 9. The question arises whether it is possible to derive better upper bounds within a general class of superhedging strategies consisting of buying/selling available stock call and put options. In order to be able to answer this question, we have to introduce the class of admissible strategies $\overline{\mathcal{I}}$.

Definition 3 (The class $\overline{\mathcal{I}}$) *The class $\overline{\mathcal{I}}$ consists of all $2n$-dimensional functions $\underline{\nu} \equiv (\nu_{1c}, \nu_{1p}, \ldots, \nu_{nc}, \nu_{np})$, of which for each i, the functions $\nu_{ic} : \mathbb{R} \to \mathbb{R}$ and $\nu_{ip} : \mathbb{R} \to \mathbb{R}$ are r.c. jump functions, only having jumps at $K_{i,j}$, $j = 0, 1, 2, \ldots, m_i + 1$. Jumps upwards as well as downwards are allowed.*

We will consider the class of investment strategies where for each stock i at current time 0, calls and/or puts can be bought (i.e. holding a long position) or sold (i.e. holding a short position). The positions taken are assumed to be held until time T, and then exercised. Any such static investment strategy can be uniquely described by a $2n$-dimensional function $\underline{\nu} \in \overline{\mathcal{I}}$, where for any stock i and any strike $y \geq 0$, we interprete $\nu_{ic}(y)$ as the number of call options purchased with a strike smaller than or equal to y. Similarly, for any stock i and any strike $y \geq 0$, the value of $\nu_{ip}(y)$ is the number of put options purchased with a strike price smaller than or equal to y.

The pay-off at time T of the investment strategy $\underline{\nu} \in \overline{\mathcal{I}}$ is given by

$$\mathsf{Pay\text{-}off}\ \left[\underline{\nu}, \underline{X}\right] = \sum_{i=1}^{n} \sum_{j=0}^{m_i+1} \left((X_i - K_{i,j})_+ \ \Delta\nu_{ic}(K_{i,j}) + (K_{i,j} - X_i)_+ \ \Delta\nu_{ip}(K_{i,j}) \right), \tag{127}$$

where $\underline{X} \equiv (X_1, X_2, \ldots, X_n)$ is the vector of the individual stock prices at time T and where $\Delta\nu_{ic}(K_{i,j})$ and $\Delta\nu_{ip}(K_{i,j})$ are the magnitudes of the jumps of the function ν_{ic} and ν_{ic} at $K_{i,j}$. The corresponding price of this investment strategy is given by

$$\mathsf{Price}\ \left[\underline{\nu}\right] = \sum_{i=1}^{n} \sum_{j=0}^{m_i+1} \left(C_i\left[K_{i,j}\right]\ \Delta\nu_{ic}(K_{i,j}) + P_i\left[K_{i,j}\right]\ \Delta\nu_{ip}(K_{i,j}) \right). \tag{128}$$

As before, we will write these sums in terms of Riemann-Stieltjes integrals. This means that we rewrite the pay-off and the price formulas as follows:

$$\mathsf{Pay\text{-}off}\ \left[\underline{\nu}, \underline{X}\right] = \sum_{i=1}^{n} \left(\int_{-\infty}^{+\infty} (X_i - y)_+ \ \mathrm{d}\nu_{ic}(y) + \int_{-\infty}^{+\infty} (y - X_i)_+ \ \mathrm{d}\nu_{ip}(y) \right) \tag{129}$$

and

$$\mathsf{Price}\ \left[\underline{\nu}\right] = \sum_{i=1}^{n} \left(\int_{-\infty}^{+\infty} C_i\left[y\right]\ \mathrm{d}\nu_{ic}(y) + \int_{-\infty}^{+\infty} P_i\left[y\right]\ \mathrm{d}\nu_{ip}(y) \right). \tag{130}$$

We are only interested in investment strategies that super-replicate the pay-off $(S-K)_+$ of the index call option or the pay-off $(K-S)_+$ of the index put option. Therefore, we define the subclass $\overline{\mathcal{C}}_K$ of super-replicating strategies for the index call option with pay-off $(S-K)_+$ and the subclass $\overline{\mathcal{P}}_K$ of super-replicating strategies for the index put option with pay-off $(K-S)_+$.

Definition 4 (The classes $\overline{\mathcal{C}}_K$ and $\overline{\mathcal{P}}_K$) *For any $K \geq 0$, the classes $\overline{\mathcal{C}}_K$ and $\overline{\mathcal{P}}_K$ are defined by*

$$\overline{\mathcal{C}}_K = \left\{ \underline{\nu} \in \overline{\mathcal{I}} \mid \left(\sum_{i=1}^{n} w_i\ x_i - K \right)_+ \leq \mathsf{Pay\text{-}off}\ \left[\underline{\nu}, \underline{x}\right]\ \text{for all } \underline{x} \right\}, \tag{131}$$

$$\overline{\mathcal{P}}_K = \left\{ \underline{\nu} \in \overline{\mathcal{I}} \mid \left(K - \sum_{i=1}^{n} w_i\ x_i \right)_+ \leq \mathsf{Pay\text{-}off}\ \left[\underline{\nu}, \underline{x}\right]\ \text{for all } \underline{x} \right\}. \tag{132}$$

In this definition, $\underline{x} \equiv (x_1, x_2, \ldots, x_n)$ and 'for all $\underline{x}$' has to be understood as

$$\text{'for all } \underline{x} \text{ with } x_i \in \left[\overline{F}_{X_i}^{-1+}(0), \overline{F}_{X_i}^{-1}(1) \right] \text{'.} \tag{133}$$

As we have from (98) that $\overline{F}_{X_i}^{-1+}(0) \leq F_{X_i}^{-1+}(0)$ and from (92) that $\overline{F}_{X_i}^{-1}(1) = F_{X_i}^{-1}(1) = K_{i,m_i+1}$, we find that the set of $\underline{x}$-values for which the inequalities in the definitions above have to hold is a support of $(X_1, X_2, \ldots, X_n)$ in the $\mathbb{Q}$-world and hence, also in the $\mathbb{P}$-world. We can conclude that

$$\mathbb{P}\left[(S-K)_+ \leq \mathsf{Pay\text{-}off}\ \left[\underline{\nu}, \underline{X}\right]\right] = 1, \qquad \text{for any } \underline{\nu} \in \overline{\mathcal{C}}_K. \tag{134}$$

A similar remark holds for any $\underline{\nu} \in \overline{\mathcal{P}}_K$.

In the following example, we show that the super-replicating strategies that were considered in Theorem 9 are elements of the class $\overline{\mathcal{C}}_K$ and $\overline{\mathcal{P}}_K$, respectively.

Example 3 (Two simple investment strategies) *Consider the investment strategy* $\underline{\nu}^* \equiv \left(\nu_{1c}^*, \nu_{1p}^*, \ldots, \nu_{nc}^*, \nu_{np}^*\right) \in \overline{\mathcal{I}}$, *where for any* $i \in N_K$, *the functions* ν_{ic}^* *and* ν_{ip}^* *are defined by*

$$\nu_{ic}^*(y) = \begin{cases} 0 & \text{if} \quad y < K_{i,j_i}, \\ w_i & \text{if} \quad y \geq K_{i,j_i}, \end{cases} \quad \text{and } \nu_{ip}^*(y) \equiv 0,$$

while for any $i \in \overline{N}_K$, *the functions* ν_{ic}^* *and* ν_{ip}^* *are defined by*

$$\nu_i^*(y) = \begin{cases} 0 & \text{if} \quad y < K_{i,j_i}, \\ \alpha_K w_i & \text{if} \quad K_{i,j_i} \leq y < K_{i,j_i+1}, \\ w_i & \text{if} \quad y \geq K_{i,j_i+1}, \end{cases} \quad \text{and } \nu_{ip}^*(y) \equiv 0.$$

Obviously, $\underline{\nu}^*$ *is the super-replicating strategy for the index call option that was considered in Theorem 9 and hence,*

$$\underline{\nu}^* \in \overline{\mathcal{C}}_K. \tag{135}$$

The pay-off of this strategy is given by:

$$\begin{aligned} \mathit{Pay\text{-}off}\ \left[\underline{\nu}^*, \underline{X}\right] = & \sum_{i \in N_K} w_i \left(X_i - K_{i,j_i}\right)_+ \\ & + \sum_{i \in \overline{N}_K} w_i \left(\alpha_K \left(X_i - K_{i,j_i}\right)_+ + (1-\alpha_K)\left(X_i - K_{i,j_i+1}\right)_+\right), \end{aligned}$$

whereas its price is given by

$$\mathit{Price}\ \left[\underline{\nu}^*\right] = \overline{C}^c\left[K\right]. \tag{136}$$

Similarly, we define the investment strategy $\underline{\eta}^* \equiv \left(\eta_{1c}^*, \eta_{1p}^*, \ldots, \eta_{nc}^*, \eta_{np}^*\right) \in \overline{\mathcal{I}}$, *where for any* $i \in N_K$, *the functions* η_{ic}^* *and* η_{ip}^* *are defined by*

$$\eta_{ic}^*(y) \equiv 0 \text{ and } \eta_{ip}^*(y) = \begin{cases} 0 & \text{if} \quad y < K_{i,j_i} \\ w_i & \text{if} \quad y \geq K_{i,j_i}, \end{cases}$$

while for any $i \in \overline{N}_K$, *the functions* ν_{ic}^* *and* ν_{ip}^* *are defined by*

$$\eta_{ic}^*(y) = 0 \text{ and } \eta_{ip}^*(y) = \begin{cases} 0 & \text{if} \quad y < K_{i,j_i}, \\ \alpha_K w_i & \text{if} \quad K_{i,j_i} \leq y < K_{i,j_i+1}, \\ w_i & \text{if} \quad y \geq K_{i,j_i+1}. \end{cases}$$

The investment strategy $\underline{\eta}^*$ is the super-replicating strategy for the index put option that was considered in Theorem 9 and hence,

$$\underline{\eta}^* \in \overline{\mathcal{P}}_K. \tag{137}$$

Its price is given by

$$\mathsf{Price}\ \left[\underline{\eta}^*\right] = \overline{P}^c\left[K\right]. \tag{138}$$

The investment strategies $\underline{\nu}^*$ and $\underline{\eta}^*$ will turn out to be optimal super-replicating strategies for the index call and put option, respectively. ∇

In the following theorem, we look for the cheapest strategy contained in $\overline{\mathcal{C}}_K$ which super-replicates the pay-off $(S-K)_+$ of the index option $C\left[K\right]$, as well as for the cheapest strategy contained in $\overline{\mathcal{P}}_K$ which super-replicates the pay-off $(K-S)_+$ of the index option $P\left[K\right]$.

Theorem 10 *Let $\underline{\nu}^* \in \overline{\mathcal{C}}_K$ and $\underline{\eta}^* \in \overline{\mathcal{P}}_K$ be the investment strategies defined in Example 3. For any $K \in \left(F^{-1+}_{\overline{S}^c}(0), F^{-1}_{\overline{S}^c}(1)\right)$ it holds that*

$$\min_{\underline{\nu}\in\overline{\mathcal{C}}_K} \mathsf{Price}\ \left[\underline{\nu}\right] = \mathsf{Price}\ \left[\underline{\nu}^*\right] = \overline{C}^c\left[K\right] \tag{139}$$

and

$$\min_{\underline{\nu}\in\overline{\mathcal{P}}_K} \mathsf{Price}\ \left[\underline{\nu}\right] = \mathsf{Price}\ \left[\underline{\eta}^*\right] = \overline{P}^c\left[K\right]. \tag{140}$$

Proof. Consider the investment strategy $\underline{\nu} \in \overline{\mathcal{C}}_K$. Replacing the x_i by $\overline{F}^{-1}_{X_i}(U)$ in the pay-off inequality

$$\left(\sum_{i=1}^n w_i\ x_i - K\right)_+ \leq \sum_{i=1}^n \left(\int_{-\infty}^{+\infty} (X_i-y)_+\ \mathrm{d}\nu_{ic}\left(y\right) + \int_{-\infty}^{+\infty} (y-X_i)_+\ \mathrm{d}\nu_{ip}\left(y\right)\right)$$

and taking expectations leads to

$$\mathbb{E}\left[\left(\overline{S}^c - K\right)_+\right] \leq \sum_{i=1}^n \left(\int_{-\infty}^{+\infty} \mathbb{E}\left[(X_i-y)_+\right]\ \mathrm{d}\nu_{ic}(y) + \int_{-\infty}^{+\infty} \mathbb{E}\left[(y-X_i)_+\right]\ \mathrm{d}\nu_{ip}(y)\right).$$

Multiplying the left and right hand side by e^{-rT}, this inequality can be rewritten as

$$\overline{C}^c\left[K\right] \leq \mathsf{Price}\ \left[\underline{\nu}\right].$$

As this inequality holds for any $\underline{\nu} \in \overline{\mathcal{C}}_K$, we can conclude that

$$\overline{C}^c\left[K\right] \leq \inf_{\underline{\nu}\in\overline{\mathcal{C}}_K} \mathsf{Price}\ \left[\underline{\nu}\right].$$

On the other hand, as $\underline{\nu}^* \in \overline{\mathcal{C}}_K$, we have that

$$\inf_{\underline{\nu} \in \overline{\mathcal{C}}_K} \mathsf{Price}\ [\underline{\nu}] \leq \mathsf{Price}\ [\underline{\nu}^*] = \overline{C}^c [K] .$$

Combining these results, we can conclude that the stated results holds true for the call option case.
The put option case can be proven in a similar way. ■

From Theorem 10, it follows that the cheapest super-replicating strategy contained in $\overline{\mathcal{C}}_K$ is given by $\underline{\nu}^*$, whereas the cheapest super-replicating strategy contained in $\overline{\mathcal{P}}_K$ is given by $\underline{\eta}^*$. The prices of these cheapest super-replicating strategies are equal to the upper bounds $\overline{C}^c [K]$ and $\overline{P}^c [K]$ that we derived in Theorem 6. Although we allow portfolios consisting of any available strike per individual stock, the cheapest of these strategies only invests in at most two strikes per stock.

Suppose for a moment that the index call option $C [K]$ is not traded in the market, but sold over-the-counter. In this case, both the seller and the buyer of this option may think of $\overline{C}^c [K]$ as a fair price for the index option. Indeed, from the seller's point of view, $\overline{C}^c [K]$ may be a reasonable price for the index call option as he can use this amount to acquire the portfolio $\underline{\nu}^*$, resulting in a pay-off which always exceeds the pay-off of the index option that he is due to the buyer. On the other hand, the buyer of the index option cannot find a cheaper super-replicating strategy in the market. In case the index option was sold over-the-counter at a higher price than its comonotonic price $\overline{C}^c [K]$, the buyer may prefer to buy the cheaper investment portfolio $\underline{\nu}^*$. A similar reasoning can be made concerning the over-the-counter index put option price.

5.4 The upper bound is the least upper bound for the index option price

As before, we use the symbol $\mathcal{D}_n$ to denote the class of all n-dimensional cdf's on the non-negative orthant of $\mathbb{R}^n$, whereas the symbols $F_i, i = 1, \ldots, n$ are used to denote the marginal cdf's of $F \in \mathcal{D}_n$. We first define the subclass $\overline{\mathcal{R}}_n$ of $\mathcal{D}_n$.

Definition 5 (The Fréchet class generated by the observed stock option prices) *The class $\overline{\mathcal{R}}_n$ of n-dimensional cdf's F is defined as*

$$\overline{\mathcal{R}}_n = \left\{ F \in \mathcal{D}_n \mid \mathrm{e}^{-rT_i} \mathbb{E}_{F_i} \left[(X_i - K_{i,j})_+ \right] = C_i [K_{i,j}] ; \quad i = 1, \ldots, n \text{ and } j = 0, 1, \ldots, m_i + 1 \right\} . \tag{141}$$

The Fréchet class $\overline{\mathcal{R}}_n$ consists of all multivariate distributions F which are consistent with the observed call option prices $C_i [K_{i,j}]$.

From the put-call parity (21), it follows that the class $\overline{\mathcal{R}}_n$ can also be seen as the class of all multivariate distributions F, which are consistent with the observed put option prices $P_i[K_{i,j}]$:

$$\overline{\mathcal{R}}_n = \left\{F \in \mathcal{D}_n \mid \mathrm{e}^{-rT_i}\mathbb{E}_{F_i}\left[(K_{i,j} - X_i)_+\right] = P_i[K_{i,j}]\,; \right.$$
$$\left. i = 1, \ldots, n \text{ and } j = 0, 1, \ldots, m_i + 1\right\}.$$

Taking into account (79) and (87), we find that the cdf of the comonotonic random vector $\left(\overline{F}_{X_1}^{-1}(U), \overline{F}_{X_2}^{-1}(U), \ldots, \overline{F}_{X_n}^{-1}(U)\right)$ with marginal distributions $\overline{F}_{X_i}$ defined in (88) is an element of $\overline{\mathcal{R}}_n$.

In the finite market case that we consider in this section, the only information that we have about the cdf of $(X_1, X_2, \ldots, X_n)$ is that it belongs to $\overline{\mathcal{R}}_n$. This information does not allow us to determine the index option prices $C[K]$ and $P[K]$, neither does it allow us to derive $C^c[K]$ and $P^c[K]$. However, this information is sufficient to specify the cdf of $\overline{S}^c$ unambigously and hence, it allows us to determine the comonotonic index option prices $\overline{C}^c[K]$ and $\overline{P}^c[K]$.

Theorem 11 (The least upper bound for the index option price) *For any* $K \geq 0$*, one has that*

$$\max_{F \in \overline{\mathcal{R}}_n} \mathrm{e}^{-rT}\mathbb{E}_F\left[(S - K)_+\right] = \overline{C}^c[K] \tag{142}$$

and

$$\max_{F \in \overline{\mathcal{R}}_n} \mathrm{e}^{-rT}\mathbb{E}_F\left[(K - S)_+\right] = \overline{P}^c[K]\,. \tag{143}$$

Moreover, in both cases the maximum is obtained for the cdf of $\left(\overline{F}_{X_1}^{-1}(U), \overline{F}_{X_2}^{-1}(U), \ldots, \overline{F}_{X_n}^{-1}(U)\right)$.

Proof. Suppose that $F \in \overline{\mathcal{R}}_n$ is the cdf of $(X_1, X_2, \ldots, X_n)$. The stop-loss relation $S \leq_{\mathrm{cx}} \overline{S}^c$ in (105) implies that

$$\mathbb{E}_F\left[(S - K)_+\right] \leq \mathbb{E}\left[\left(\overline{S}^c - K\right)_+\right]$$

As this inequality holds for any $F \in \overline{\mathcal{R}}_n$, we find that

$$\sup_{F \in \overline{\mathcal{R}}_n} \mathbb{E}_F\left[(S - K)_+\right] \leq \mathbb{E}\left[\left(\overline{S}^c - K\right)_+\right].$$

On the other hand, from the fact that the cdf of $\left(\overline{F}_{X_1}^{-1}(U), \overline{F}_{X_2}^{-1}(U), \ldots, \overline{F}_{X_n}^{-1}(U)\right)$ is an element of $\overline{\mathcal{R}}_n$, we also have that

$$\mathbb{E}[(\overline{S}^c - K)_+] \leq \sup_{F \in \overline{\mathcal{R}}_n} \mathbb{E}_F[(S - K)_+].$$

Combining these observations leads to (142).
The put option case is proven in a similar way.

■

Theorem 11 states that both the upper bounds $\overline{C}^c[K]$ and $\overline{P}^c[K]$ that we derived in Theorem 6 can be interpreted as least upper bounds in the sense that they correspond to the largest possible expected discounted pay-off of the corresponding index option, given the limited information about the marginal distributions of the underlying stocks that is contained in the observed stock option prices. Somewhat loosely speaking, $\overline{C}^c[K]$ is the lowest upper bound for the index call option price with strike K in the class of all models which are consistent with the observed stock option prices. A similar interpretation holds true for the upper bound $\overline{P}^c[K]$.

The upper bound $\overline{C}^c[K]$, resp. $\overline{P}^c[K]$, coincides with the index option price $C[K]$, resp. $P[K]$, in case the risk-neutral multivariate distribution of the price vector $\underline{X}$ is equal to the distribution of $\left(\overline{F}_{X_1}^{-1}(U), \overline{F}_{X_2}^{-1}(U), \ldots, \overline{F}_{X_n}^{-1}(U)\right)$. The question whether this upper bound is reachable in the sense that it is possible to construct an arbitrage-free market with the observed index and stock option prices and with this risk-neutral multivariate pricing distribution is considered in Hobson et al. (2005).

5.5 Computational and practical aspects

In this section we will consider several computational aspects related to the finite market case. We first prove the following lemma, which will be needed hereafter. The notations used in this section correspond to the notations introduced before.

Lemma 5 *Consider a real number x which can be expressed as*

$$x = \sum_{i=1}^{n} w_i x_i \tag{144}$$

with $\underline{x} = (x_1, x_2 \ldots, x_n)$ being an element of a comonotonic support of $\left(\overline{F}_{X_1}^{-1}(U), \overline{F}_{X_2}^{-1}(U), \ldots, \overline{F}_{X_n}^{-1}(U)\right)$. Then one has that

$$F_{\overline{S}^c}(x) = \min_{i \in \{1,2,\ldots,n\}} \overline{F}_{X_i}(x_i). \tag{145}$$

Proof. Any two elements $\underline{x}$ and $\underline{y}$ of a given comonotonic support of $\left(\overline{F}_{X_1}^{-1}(U), \overline{F}_{X_2}^{-1}(U), \ldots, \overline{F}_{X_n}^{-1}(U)\right)$ are ordered componentwise. Hence either $x_i \leq y_i$ must hold for all i or $y_i \leq x_i$ must hold for all i. Let x be defined by (144), then the componentwise ordering of $\underline{x}$ and $\underline{y}$ leads to

$$\sum_{i=1}^{n} w_i y_i \leq x \Longleftrightarrow y_i \leq x_i, \text{ for all } i = 1, 2, \ldots, n, \tag{146}$$

or, equivalently,

$$\mathbb{I}\left(\sum_{i=1}^{n} w_i y_i \leq x\right) = \mathbb{I}\left(y_1 \leq x_1, y_2 \leq x_2, \ldots, y_n \leq x_n\right), \tag{147}$$

where the notation $\mathbb{I}(A)$ is used to denote the indicator function which equals 1 when A holds true and 0 otherwise. Replacing each y_i by $\overline{F}_{X_i}^{-1}(U)$ in (147) and taking expectations on both sides leads to

$$\begin{aligned} F_{\overline{S}^c}(x) &= \mathbb{P}\left[\overline{F}_{X_i}^{-1}(U) \leq x_1, \overline{F}_{X_2}^{-1}(U) \leq x_2, \ldots, \overline{F}_{X_n}^{-1}(U) \leq x_n\right] \\ &= \min_{i \in \{1,2,\ldots,n\}} \overline{F}_{X_i}(x_i), \end{aligned}$$

where in the last step we made use of the well-known expression for the multivariate cdf of a comonotonic vector. ■

5.5.1 Numerical evaluation of the upper bounds for $C[K]$ and $P[K]$

In this subsection, we explain how to determine the comonotonic index option prices $\overline{C}^c[K]$ and $\overline{P}^c[K]$ as well as the corresponding super-replicating strategies $\underline{\nu}^*$ and $\underline{\eta}^*$ for the index options $C[K]$ and $P[K]$, respectively.

Starting from either the observed stock option prices $C_i[K_{i,j}]$ or the observed stock put options $P_i[K_{i,j}]$, we can determine all $\overline{F}_{X_i}(K_{i,j})$ from Lemma 3. In a second step, the extreme outcomes $F_{\overline{S}^c}^{-1+}(0)$ and $F_{\overline{S}^c}^{-1}(1)$ are obtained from

$$F_{\overline{S}^c}^{-1+}(0) = \sum_{i=1}^{n} w_i \overline{F}_{X_i}^{-1+}(0) \text{ with } \overline{F}_{X_i}^{-1+}(0) = \min_{j \in \{0,1,\ldots,m_i\}} \left\{K_{i,j} \mid \overline{F}_{X_i}(K_{i,j}) > 0\right\}$$

and

$$F_{\overline{S}^c}^{-1}(1) = \sum_{i=1}^{n} w_i K_{i,m_i+1}.$$

For $K \notin \left(F_{\overline{S}^c}^{-1+}(0), F_{\overline{S}^c}^{-1}(1)\right)$, the comonotonic index option prices $\overline{C}^c[K]$ and $\overline{P}^c[K]$ are equal to the exact index option prices:

$$\begin{aligned} C[K] &= 0, \qquad K \geq F_{\overline{S}^c}^{-1}(1), \\ P[K] &= 0, \qquad K \leq F_{\overline{S}^c}^{-1+}(0), \end{aligned}$$

and

$$\begin{aligned} C[K] &= e^{-rT}\left(\mathbb{E}[S] - K\right), \qquad K \leq F_{\overline{S}^c}^{-1+}(0), \\ P[K] &= e^{-rT}\left(K - \mathbb{E}[S]\right), \qquad K \geq F_{\overline{S}^c}^{-1}(1). \end{aligned}$$

In the remainder of this section, we assume that $K \in \left(F_{\overline{S}^c}^{-1+}(0), F_{\overline{S}^c}^{-1}(1)\right)$, except if explicitely stated otherwise. This assumption implies that $F_{\overline{S}^c}(K) \in (0,1)$.

In order to be able to determine the upper bounds $\overline{C}^c[K]$ and $\overline{P}^c[K]$ for the index call and put options $C[K]$ and $P[K]$, one has to determine $F_{\overline{S}^c}(K)$, the indices j_i, $i = 1, 2, \ldots, n$, the corresponding sets N_K and $\overline{N}_K$, and also the coefficient α_K. Let us first consider the problem of determining $F_{\overline{S}^c}(K)$.

Lemma 6 *For any* $K \in \left(F_{\overline{S}^c}^{-1+}(0), F_{\overline{S}^c}^{-1}(1)\right)$ *one has that*

$$F_{\overline{S}^c}(K) = \min_{i \in \{1,\ldots,n\}} \left\{ \overline{F}_{X_i}\left(K_{i,j_i}\right) \right\}. \tag{148}$$

Proof. First notice that

$$F_{\overline{S}^c}\left(K\right) = F_{\overline{S}^c}\left(F_{\overline{S}^c}^{-1}\left(F_{\overline{S}^c}\left(K\right)\right)\right)$$

As $F_{\overline{S}^c}^{-1}\left(F_{\overline{S}^c}\left(K\right)\right) = \sum_{i=1}^n w_i \overline{F}_{X_i}^{-1}\left(F_{\overline{S}^c}\left(K\right)\right)$ and $\left(\overline{F}_{X_1}^{-1}\left(F_{\overline{S}^c}\left(K\right)\right), \ldots, \overline{F}_{X_n}^{-1}\left(F_{\overline{S}^c}\left(K\right)\right)\right)$ is an element of a comonotonic support of $\left(\overline{F}_{X_1}^{-1}\left(U\right), \ldots, \overline{F}_{X_n}^{-1}\left(U\right)\right)$, a direct application of Lemma 5 leads to

$$F_{\overline{S}^c}\left(K\right) = \min_{i \in \{1,2,\ldots,n\}} \overline{F}_{X_i}(\overline{F}_{X_i}^{-1}\left(F_{\overline{S}^c}\left(K\right)\right))$$

Expression (148) follows then from (120). ■

Unfortunately, Lemma 6 does not provide us with a straightforward way for determining $F_{\overline{S}^c}(K)$. Indeed, the j_i depend on the value of $F_{\overline{S}^c}(K)$ and hence, (148) only gives an implicit expression for $F_{\overline{S}^c}(K)$. In the following theorem we present an explicit expression for $F_{\overline{S}^c}(K)$. The proof of the theorem makes use of Lemma 6.

Theorem 12 *Let* A *be the set defined by*

$$A = \left\{ \overline{F}_{X_i}\left(K_{i,j}\right) \mid i = 1, \ldots, n \text{ and } j = 0, 1, \ldots, m_i \right\} \setminus \{0\}. \tag{149}$$

For any $K \in \left(F_{\overline{S}^c}^{-1+}(0), F_{\overline{S}^c}^{-1}(1)\right)$, *the value of* $F_{\overline{S}^c}(K)$ *follows from*

$$F_{\overline{S}^c}(K) = \max\left\{ p \in A \mid \sum_{i=1}^n w_i \overline{F}_{X_i}^{-1}\left(p\right) \leq K \right\}. \tag{150}$$

Proof. From Lemma 6 it follows immediately that $F_{\overline{S}^c}(K)$ is equal to one of the $\overline{F}_{X_i}\left(K_{i,j}\right)$, $i = 1, \ldots, n$; $j = 0, 1, \ldots, m_i{+}1$. Furthermore, for any $K \in \left(F_{\overline{S}^c}^{-1+}(0), F_{\overline{S}^c}^{-1}(1)\right)$ one has that $0 < F_{\overline{S}^c}(K) < 1$. These observations imply that

$$F_{\overline{S}^c}(K) \in A.$$

Obviously it holds that

$$F_{\overline{S}^c}(K) = \max\left\{ p \in A \mid p \leq F_{\overline{S}^c}(K) \right\}.$$

Taking into account (8), this relation can be transformed into

$$F_{\overline{S}^c}(K) = \max\left\{ p \in A \mid F_{\overline{S}^c}^{-1}(p) \leq K \right\}.$$

Combining this expression with the additivity property for the quantiles of a comonotonic sum leads to (150). ■

Combining the previous results and Theorem 12 allows us to write down the following algorithm for determining $F_{\overline{S}^c}(K)$.

Algorithm 1 (Determining $F_{\overline{S}^c}(K)$) *For any* $K \in \left(F_{\overline{S}^c}^{-1+}(0), F_{\overline{S}^c}^{-1}(1)\right)$, *determine* $F_{\overline{S}^c}(K)$ *as follows:*

1. Calculate all elements $\overline{F}_{X_i}(K_{i,j})$ of the set A defined in (149):
   For any $i = 1, \ldots, n$ and $j = 0, 1, \ldots, m_i$, we have that

$$\overline{F}_{X_i}(K_{i,j}) = 1 + e^{rT} \frac{C_i[K_{i,j+1}] - C_i[K_{i,j}]}{K_{i,j+1} - K_{i,j}},$$

   or, equivalently,

$$\overline{F}_{X_i}(K_{i,j}) = e^{rT} \frac{P_i[K_{i,j+1}] - P_i[K_{i,j}]}{K_{i,j+1} - K_{i,j}}.$$

2. Calculate $\overline{F}_{X_i}^{-1}(p)$ for any $i = 1, \ldots, n$ and any $p \in A$:

$$\overline{F}_{X_i}^{-1}(p) = K_{i,j} \text{ if } \overline{F}_{X_i}(K_{i,j-1}) < p \leq \overline{F}_{X_i}(K_{i,j}), \qquad j = 0, 1, \ldots, m_i + 1.$$

3. Calculate $F_{\overline{S}^c}(K)$ from (150):

$$F_{\overline{S}^c}(K) = \max\left\{p \in A \mid \sum_{i=1}^{n} w_i \overline{F}_{X_i}^{-1}(p) \leq K\right\}$$

In case $K \notin \left(F_{\overline{S}^c}^{-1+}(0), F_{\overline{S}^c}^{-1}(1)\right)$, it is straightforward to determine $F_{\overline{S}^c}(K)$. Indeed,

$$F_{\overline{S}^c}(K) = 1 \text{ if } K \geq F_{\overline{S}^c}^{-1}(1),$$

whereas

$$F_{\overline{S}^c}(K) = 0 \text{ if } K < F_{\overline{S}^c}^{-1+}(0)$$

and

$$F_{\overline{S}^c}\left(F_{\overline{S}^c}^{-1+}(0)\right) = \min_{i \in \{1,\ldots,n\}} \overline{F}_{X_i}\left(\overline{F}_{X_i}^{-1+}(0)\right) \tag{151}$$

This last expression follows from Lemma 5.

A computationally better but more complicated algorithm for determining $F_{\overline{S}^c}(K)$ is described in Chen et al. (2008). After having calculated $F_{\overline{S}^c}(K)$, the indices $j_i, i = 1, 2, \ldots, n$, can be determined from (114), while the sets N_K and $\overline{N}_K$ are given by (115) and (116), respectively. From (111), together with (112) and (120) we find that

$$K = \sum_{i \in N_K} w_i K_{i,j_i} + \sum_{i \in \overline{N}_K} w_i \left(\alpha_K K_{i,j_i} + (1 - \alpha_K) K_{i,j_i+1}\right). \tag{152}$$

Solving this equation for α_K leads to

$$\alpha_K = 1 - \frac{K - \sum_{i=1}^{n} w_i K_{i,j_i}}{\sum_{i \in \overline{N}_K} w_i \left(K_{i,j_i+1} - K_{i,j_i}\right)} \tag{153}$$

From Lemma 6 it follows that the set $\overline{N}_K$ contains at least one element. Furthermore, recall that we assumed that all K_{i,m_i+1} are finite. Both observations guarantee that α_K is well-defined.

In practice, it may happen that the chain of traded strikes is different for the stock calls and puts. In order to explain how to cope with this problem, let us suppose that $P_i[K]$ is traded, while $C_i[K]$ is not. As long as there is at least one strike K_i^* for which the prices $C_i[K^*]$ and $P[K_i^*]$ are traded, the forward price $\mathbb{E}[X_i]$ can be calculated in a model-free way from the put-call parity applied to this couple:

$$e^{-rT}\mathbb{E}[X_i] = C_i[K^*] - P_i[K^*] + e^{-rT}K^*.$$

The pay-off of receiving X_i at time T can be replicated by an investment strategy consisting of buying the call $C_i[K^*]$, selling the put $P_i[K^*]$ and putting $e^{-rT}K^*$ on the bank account. The price $C_i[K]$ of the non-traded call option can then be backed out of the traded put option $P_i[K]$ with the help of the put-call parity:

$$C_i[K] = P_i[K] + e^{-rT}\mathbb{E}[X_i] - e^{-rT}K.$$

In this case, a long position in the non-traded call option $C_i[K]$ has to be understood as a long position in the traded put option $P_i[K]$, receiving the stock i at time T and borrowing $e^{-rT}K$.

5.5.2 On the choice of the maximal values K_{i,m_i+1}

In this subsection we make the following assumption concerning the choice of the 'maximal' values K_{i,m_i+1} of the stock prices X_i:

$$\max_{i\in\{1,\ldots,n\}} \overline{F}_{X_i}(K_{i,m_i-1}) < \min_{i\in\{1,\ldots,n\}} \overline{F}_{X_i}(K_{i,m_i}). \tag{154}$$

Notice that we implicitely assume here that all $m_i > 0$. From (88) we find that any $\overline{F}_{X_i}(K_{i,m_i})$ is given by

$$\overline{F}_{X_i}(K_{i,m_i}) = 1 - e^{rT}\frac{C_i[K_{i,m_i}]}{K_{i,m_i+1} - K_{i,m_i}}.$$

This implies that $\overline{F}_{X_i}(K_{i,m_i})$ is an increasing function of K_{i,m_i+1} and by choosing K_{i,m_i+1} sufficiently large, the value of $\overline{F}_{X_i}(K_{i,m_i})$ can be made as close to 1 as desired. This means that the assumption (154) will be fulfilled, provided all K_{i,m_i+1} are chosen sufficiently large.
Taking into account (88), we can rewrite the assumption (154) as follows:

$$K_{i,m_i+1} > K_{i,m_i} + C_i[K_{i,m_i}] \times \max_{j\in\{1,\ldots,n\}}\left\{\frac{K_{j,m_j} - K_{j,m_j-1}}{C_j\left[K_{j,m_j-1}\right] - C_j\left[K_{j,m_j}\right]}\right\}, \qquad i = 1, 2, \ldots, n. \tag{155}$$

Throughout this subsection, we will also silently assume that $\overline{F}_{X_i}^{-1+}(0) < K_{i,m_i}$ for any $i = 1, \ldots, n$. This assumption implies that $\overline{F}_{X_i}(K_{i,m_i}) > 0$ for all i.

Lemma 7 *Assume that the K_{i,m_i+1}, $i=1,\ldots,n$, are sufficiently large in the sense that (154) holds. Then one has that*

$$F_{\overline{S}^c}(K) < \min_{i\in\{1,\ldots,n\}} \overline{F}_{X_i}\left(K_{i,m_i}\right), \qquad \textit{if } K < \sum_{i=1}^{n} w_i K_{i,m_i}, \tag{156}$$

while

$$F_{\overline{S}^c}\left(\sum_{i=1}^{n} w_i K_{i,m_i}\right) = \min_{i\in\{1,\ldots,n\}} \overline{F}_{X_i}\left(K_{i,m_i}\right). \tag{157}$$

Proof. (a) We first prove that (157) holds.
Let $p_{\min}$ be defined by

$$p_{\min} = \min_{i\in\{1,\ldots,n\}} \overline{F}_{X_i}\left(K_{i,m_i}\right).$$

Notice that $0 < p_{\min} < 1$. From assumption (154) we find that

$$\overline{F}_{X_i}\left(K_{i,m_i-1}\right) < p_{\min} \leq \overline{F}_{X_i}\left(K_{i,m_i}\right), \qquad i=1,\ldots,n. \tag{158}$$

This implies that

$$\overline{F}_{X_i}^{-1}(p_{\min}) = K_{i,m_i}, \qquad i=1,\ldots,n.$$

Hence,

$$\sum_{i=1}^{n} w_i K_{i,m_i} = \sum_{i=1}^{n} w_i \overline{F}_{X_i}^{-1}(p_{\min}).$$

These last two expressions combined with Lemma 5 imply that (157) holds true.
(b) The r.v. $\overline{S}^c$ is a comonotonic sum of discrete r.v.'s and hence, also has a discrete distribution. As $F_{\overline{S}^c}^{-1}(p_{\min}) = \sum_{i=1}^{n} w_i \overline{F}_{X_i}^{-1}(p_{\min}) = \sum_{i=1}^{n} w_i K_{i,m_i}$we have that $\overline{S}^c$ has a strictly positive probability mass at $\sum_{i=1}^{n} w_i K_{i,m_i}$. This observation implies (156). ■

Based on the results derived above, we are now able to prove that, under rather general conditions, the upper bounds for the index option prices will not depend on the particular choices for the values of the K_{i,m_i+1}.

Theorem 13 *Assume that the K_{i,m_i+1}, $i=1,\ldots,n$, are chosen sufficiently large in the sense that (154) holds. In this case, we have that for any $K \in \left(\overline{F}_{\overline{S}^c}^{-1+}(0), \sum_{i=1}^{n} w_i K_{i,m_i}\right]$ the upper bounds $\overline{C}^c[K]$ and $\overline{P}^c[K]$ for the index option prices $C[K]$ and $P[K]$ and also the corresponding super-replicating strategies do not depend on the particular choices for the values of the K_{i,m_i+1}.*

Proof. (a) When $K < \sum_{i=1}^{n} w_i K_{i,m_i}$, we know from Lemma 7 that

$$F_{\overline{S}^c}(K) < \overline{F}_{X_i}\left(K_{i,m_i}\right), \qquad i=1,\ldots,n.$$

From the definitions of the j_i it follows then that for each $i \in \{1,\ldots,n\}$ either '$j_i = m_i$ and $i \in N_K$' or '$j_i \leq m_i - 1$' holds. Hence, the indices j_i as well as the sets N_K and

$\overline{N}_K$ do not depend on the choices of the values of the K_{i,m_i+1}. From (153) it follows that also α_K does not depend on the choice of the K_{i,m_i+1}. We can conclude that the upper bounds $\overline{C}^c[K]$ and $\overline{P}^c[K]$ for the index option prices $C[K]$ and $P[K]$ and the associated super-replicating strategies do not depend on the particular choice of the K_{i,m_i+1}.

(b) Let us now consider the limiting case where $K = \sum_{i=1}^{n} w_i K_{i,m_i}$. From Lemma 7, we know that $F_{\overline{S}^c}$ has a positive jump at K. Taking into account the definition (113) of α_K, we find that $\alpha_K = 1$ in this case. Furthermore, from assumption (154) and expression (157) we find that

$$\overline{F}_{X_i}(K_{i,m_i-1}) < F_{\overline{S}^c}(K) \leq \overline{F}_{X_i}(K_{i,m_i}), \qquad i = 1, \ldots, n,$$

which implies that $j_i = m_i$ for any $i = 1, \ldots, n$ in this case. Hence, we can again conclude that the upper bounds $\overline{C}^c[K]$ and $\overline{P}^c[K]$, as well as the associated optimal super-replicating strategies, do not depend on the particular choice of the K_{i,m_i+1}. ■

From the previous lemma, we know that under reasonable assumptions the bounds $\overline{C}^c[K]$ and $\overline{P}^c[K]$, as well as the corresponding super-replicating strategies, do not depend on the particular choices of the values of the K_{i,m_i+1}.

5.5.3 Determining the stock option prices $C_i[0]$ and $P_i[K_{i,m_i+1}]$

Until here we assumed that the stock option prices

$$C_i[0] = e^{-rT}\mathbb{E}[X_i] \tag{159}$$

and

$$P_i[K_{i,m_i+1}] = e^{-rT}(K_{i,m_i+1} - \mathbb{E}[X_i]) \tag{160}$$

are known for any stock i. In practice however, these options are not traded and hence, their prices cannot be observed. Hereafter, we will explain how to derive these prices, or equivalently, how to determine the forward prices $\mathbb{E}[X_i]$, from information that is available in the market. Notice from the previous subsection that under appropriate choices for the K_{i,m_i+1}, our results will not depend on the explicit values of these K_{i,m_i+1} and hence, knowledge of the $P_i[K_{i,m_i+1}]$ will not be required. Nevertheless, for reasons of completeness hereafter we will not only explain how to cope with the problem of unobserved values of the option prices $C_i[0]$, but we will also have a look at the problem of unknown values of $P_i[K_{i,m_i+1}]$.

Let us first consider the simple case where stock i pays no dividends in the period $[0,T]$. In this case, 'buying the call $C_i[0]$ at time 0 and holding it until maturity T' or 'buying the stock i at time 0 and selling it at time T' leads to the same pay-off X_i at time T. A no-arbitrage argument leads to the conclusion that the call option price $C_i[0]$ is equal to the spot price of the underlying stock in this case:

$$C_i[0] = X_i(0), \quad \text{for a non-dividend-paying stock.} \tag{161}$$

From (159) and (160) it follows that the put option price $P_i[K_{i,m_i+1}]$ is then given by

$$P_i[K_{i,m_i+1}] = e^{-rT}K_{i,m_i+1} - X_i(0), \quad \text{for a non-dividend-paying stock.} \tag{162}$$

Let us now consider the general case where stock i may or may not pay dividends in the period $[0, T]$. In order to determine the value of $C_i[0]$ in this case, let y be any strike for which calls and puts with expiration date T are traded on stock i. From the put-call parity (21) we find that

$$C_i[0] = C_i[y] - P_i[y] + ye^{-rT}. \tag{163}$$

Taking into account (159) and (160), this expression leads to

$$P_i[K_{i,m_i+1}] = P_i[y] - C_i[y] + e^{-rT}(K_{i,m_i+1} - y). \tag{164}$$

The relations (163) and (164) express the option prices $C_i[0]$ and $P_i[K_{i,m_i+1}]$ in terms of observed prices. Furthermore, it is straightforward to prove that the pay-offs of both $C_i[0]$ and $P_i[K_{i,m_i+1}]$ can be replicated by the pay-offs of time-0 strategies consisting of trading the stock options $C_i[y]$ and $P_i[y]$ and investing in the risk-free T-year zero coupon bond, the prices of which are given by the right hand sides of (163) and (164), respectively.

In principle $C_i[0]$ and $P_i[K_{i,m_i+1}]$ can be determined from (163) and (164) for any traded strike y. However, in order to guarantee that the strike y is sufficiently traded, in practice one often prefers to choose the strike y as close as possible to $\mathbb{E}[X_i]$, or equivalently, to choose y such that $|C_i[y] - P_i[y]|$ is as small as possible. This means that the strike $y \in \{K_{i,j} \mid j = 0, 1, \ldots, m_i + 1\}$ that is used to determine $C_i[0]$ and $P_i[K_{i,m_i+1}]$ is defined as follows:

$$y = \underset{j \in \{0,1,\ldots,m_i+1\}}{\arg\min} |C_i[K_{i,j}] - P_i[K_{i,j}]|. \tag{165}$$

Having determined the values of $C_i[0]$ or $P_i[K_{i,m_i+1}]$, we immediately find the corresponding value of $\mathbb{E}[X_i]$.

A first way to circumvent the problem that $C_i[0]$ and $P_i[K_{i,m_i+1}]$ are not directly observed is to consider strategies that replicate the pay-offs of these options. This procedure was explained above. A second way of solving this problem consists of determining upper bounds for the unobservable option prices $C_i[0]$ and $P_i[K_{i,m_i+1}]$. This procedure is considered hereafter.

A first upper bound that we considered already for $C_i[0]$ is given by

$$C_i[0] \leq X_i(0), \tag{166}$$

where the inequality can be replaced by an equality in case stock i pays no dividends. Another upper bound can be constructed for $C_i[0]$ by considering its largest possible value $C_i^M[0]$ which is consistent with the available information. Taking into account that $C_i[y]$ is convex and decreasing, that $(K_{i,1}, C_i[K_{i,1}])$ is an element of the curve $(y, C_i[y])$ and that the slope of $C_i[y]$ is given by $-e^{-rT}$ for $y \leq 0$, we find that

$$C_i[y] \leq C_i^M[y], \qquad \text{for all } y \leq K_{i,1},$$

with $C_i^M[y]$ defined by

$$C_i^M[y] = C_i[K_{i,1}] - e^{-rT}(y - K_{i,1}).$$

In particular, we find that the maximal value for $C_i[0]$ is given by

$$C_i^M[0] = C_i[K_{i,1}] + e^{-rT} K_{i,1}.$$

Hence, we find the following upper bound for the zero-strike call option:

$$C_i[0] \leq C_i[K_{i,1}] + e^{-rT} K_{i,1}, \tag{167}$$

see Figure 7. This upper bound for $C_i[0]$ is the price of a super-replicating strategy for the pay-off X_i at time T, consisting of buying the call $C_i[K_{i,1}]$ and investing $e^{-rT}K_{i,1}$ in the risk-free zero coupon bond at time 0.

In a similar way, we can derive an upper bound for the unobserved option price $P_i[K_{i,m_i+1}]$ by considering its largest possible value $P_i^M[K_{i,m_i+1}]$ which is consistent with the available information. Taking into account that $P_i[y]$ is convex and increasing, that $(K_{i,m_i}, P_i[K_{i,m_i}])$ is an element of the curve $(y, P_i[y])$ and that the slope of $P_i[y]$ is equal to e^{-rT} for $y \geq K_{i,m_i+1}$, we find that

$$P_i[y] \leq P_i^M[y], \qquad \text{for all } y \geq K_{i,m_i},$$

with $P_i^M[y]$ defined by

$$P_i^M[y] = P_i[K_{i,m_i}] + e^{-rT}(y - K_{i,m_i}).$$

In particular, we find that the maximal value for $P_i[K_{i,m_i+1}]$ is given by

$$P_i^M[K_{i,m_i+1}] = P_i[K_{i,m_i}] + e^{-rT}(K_{i,m_i+1} - K_{i,m_i}).$$

Hence, we find the following upper bound:

$$P_i[K_{i,m_i+1}] \leq P_i[K_{i,m_i}] + e^{-rT}(K_{i,m_i+1} - K_{i,m_i}). \tag{168}$$

This upper bound for $P_i[K_{i,m_i+1}]$ is the price of a super-replicating strategy for the pay-off $(K_{i,m_i+1} - X_i)$ at time T, consisting of buying the put $P_i[K_{i,m_i}]$ and investing $e^{-rT}(K_{i,m_i+1} - K_{i,m_i})$ in the risk-free zero coupon bond at time 0.

Most of the results that we derived concerning upper bounds for index options in the finite market case remain to hold if we replace $C_i[0]$ and $P_i[K_{i,m_i+1}]$ by $C_i^M[0]$ and $P_i^M[K_{i,m_i+1}]$, respectively, and replace 'buying the call $C_i[0]$' by 'buying $C_i[K_{i,1}]$ and investing $e^{-rT}K_{i,1}$in the risk-free zero coupon bond', while replacing 'buying the put $P_i[K_{i,m_i+1}]$' by 'buying the put $P_i[K_{i,m_i}]$ and investing $e^{-rT}(K_{i,m_i+1} - K_{i,m_i})$ in the risk-free zero coupon bond at time 0.

Notice that replacing $C_i[0]$ by $X_i(0)$ and 'buying $C_i[0]$' by 'buying $X_i(0)$' in all our previous results will not always be appropriate as it may lead to a non-convex call option curve. This situation will arise when $X_i(0) > C_i^M[0]$. This inequality will be fulfilled in particular when a substantial part of the stock price $X_i(0)$ consists of the market price for the future dividend payments in the period $[0, T]$.

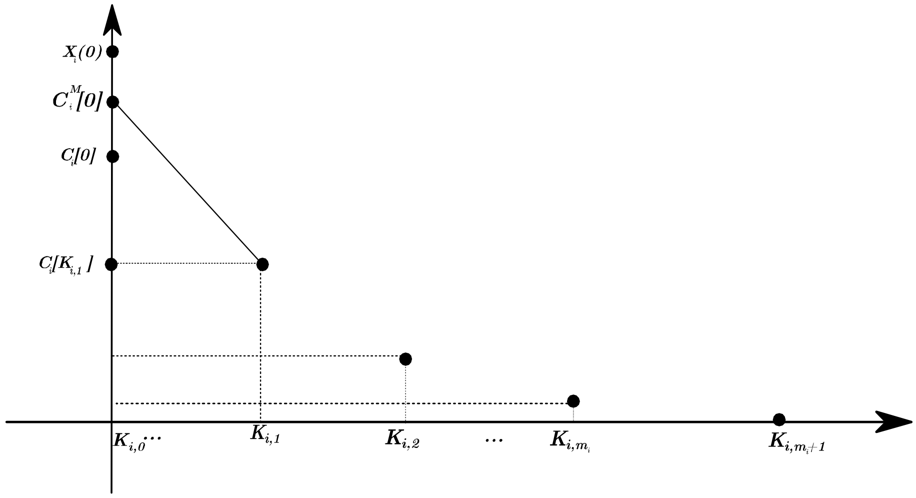

Figure 7: Upper bounds for the call price $C_i\,[0]$.

5.5.4 The upper bounds in terms of the inverses $\overline{F}_{X_i}^{-1}$

From Theorem 7 it follows that the comonotonic index option prices $\overline{C}^c\,[K]$ and $\overline{P}^c\,[K]$ are expressed in terms of the inverses $F_{\overline{X}_i}^{-1(\alpha_K)}$ with α_K defined in (153). In the following corollary, we show that these upper bounds can also be expressed in terms of the usual inverses $F_{\overline{X}_i}^{-1}$.

Corollary 2 *For any $K \in \left(F_{\overline{S}^c}^{-1+}(0), F_{\overline{S}^c}^{-1}(1)\right)$, one has that*

$$\overline{C}^c\,[K] = \sum_{i=1}^{n} w_i\; C_i\,[K_{i,j_i}] - e^{-rT}\left(K - \sum_{i=1}^{n} w_i\; K_{i,j_i}\right)(1 - F_{\overline{S}^c}(K)) \tag{169}$$

and

$$\overline{P}^c\,[K] = \sum_{i=1}^{n} w_i\; P_i\,[K_{i,j_i}] + e^{-rT}\left(K - \sum_{i=1}^{n} w_i\; K_{i,j_i}\right)F_{\overline{S}^c}(K), \tag{170}$$

where the indices j_i are defined as before.

Proof. From Corollary 1 applied to the vector $\left(\overline{F}_{X_1}^{-1}(U), \overline{F}_{X_2}^{-1}(U), \ldots, \overline{F}_{X_n}^{-1}(U)\right)$ we find that $\overline{C}^c\,[K]$ can be expressed as

$$\begin{aligned}\overline{C}^c\,[K] = &\sum_{i=1}^{n} w_i \overline{C}_i\left[\overline{F}_{X_i}^{-1}\left(F_{\overline{S}^c}(K)\right)\right]\\ &- e^{-rT}\left(K - F_{\overline{S}^c}^{-1}\left(F_{\overline{S}^c}(K)\right)\right)(1 - F_{\overline{S}^c}(K)).\end{aligned}$$

From (121) for $\alpha = 1$, we find that $\overline{C}_i\left[\overline{F}_{X_i}^{-1}\left(F_{\overline{S}^c}(K)\right)\right] = C_i\,[K_{i,j_i}]$. Furthermore, from the additivity property for quantiles of a comonotonic sum and from (119) for $\alpha = 1$, we have that $F^{-1}_{\overline{S}^c}\left(F_{\overline{S}^c}(K)\right)=\sum_{i=1}^n w_i\ K_{i,j_i}$. Combining these observations leads to (169). The expression (170) for $\overline{P}^c[K]$ can be proven in a similar way. ■

From (152) it follows that $\sum_{i=1}^n w_i K_{i,j_i}\leq K$. This implies that the second term in the right hand side of (170) is non-negative. Hence, we find that

$$\overline{C}^c[K]\leq\sum_{i=1}^n w_i C_i\left[K_{i,j_i}\right] \tag{171}$$

and $\sum_{i=1}^n w_i\ C_i\left[K_{i,j_i}\right]$ is also an upper bound for the index option price $C[K]$, although it is not the optimal one in the sense that the time-0 price of the portfolio of stock options $C_i\left[K_{i,j_i}\right]$ will not be the price of the cheapest super-replicating strategy for the index call option $C[K]$.

5.5.5 The case when no option data are available for some stocks

Recall that for any stock i, the chain of traded strikes is given by

$$0=K_{i,0}<K_{i,1}<K_{i,2}<\cdots<K_{i,m_i}<K_{i,m_i+1}=F^{-1}_{X_i}(1)<\infty.$$

In practice, there may be stocks i for which $m_i=0$, which means that there is no couple $\left(C_i\left[K_{i,j}\right],P_i\left[K_{i,j}\right]\right)$ of option prices available with $0<K_{i,j}<K_{i,m_i+1}$. This situation will occur in particular when there are only options available on a subset of the constituent stocks of the index. An example in that respect is the S&P 500 index. Hereafter, we show how the calculations of the index option upper bounds can be simplified in this case, provided the K_{i,m_i+1} fulfill an appropriate condition. Without lack of generality, throughout this section we assume that $m_i>0$ for stocks $i=1,2,\ldots,k$, while $m_i=0$ for $i=k+1,\ldots,n$.

The discrete distributions $\overline{F}_{X_i}$ of the stocks $i=k+1,\ldots,n$, follow from Lemma 3:

$$\overline{F}_{X_i}(x)=\begin{cases}0 & x<0,\\ 1-e^{rT}\dfrac{C_i[0]}{K_{i,1}} & 0\leq x<K_{i,1},\\ 1 & x\geq K_{i,1}.\end{cases} \tag{172}$$

As before, we assume that no marginal distribution $\overline{F}_{X_i}$ is a one-point distribution, see (93). For the stocks $i=k+1,\ldots,n$, this implies that $0<\overline{F}_{X_i}(0)<1$.

We introduce the notation $\overline{S}^c_k$ for the comonotonic sum of the first k stocks:

$$\overline{S}^c_k=w_1\overline{F}^{-1}_{X_1}(U)+w_2\overline{F}^{-1}_{X_2}(U)+\cdots+w_k\overline{F}^{-1}_{X_k}(U). \tag{173}$$

Furthermore, we use the symbol $\overline{C}^c_k[K]$ and $\overline{P}^c_k[K]$ to denote the corresponding comonotonic call and put option prices:

$$\overline{C}^c_k[K]=e^{-rT}\mathbb{E}\left[\left(\overline{S}^c_k-K\right)_+\right] \tag{174}$$

and

$$\overline{P}_k^c[K] = e^{-rT}\mathbb{E}\left[\left(K - \overline{S}_k^c\right)_+\right]. \tag{175}$$

For any $K \in \left(F_{\overline{S}_k^c}^{-1+}(0), F_{\overline{S}_k^c}^{-1}(1)\right)$, the value of $F_{\overline{S}_k^c}(K)$ follows from algorithm 1, where we replace n, A and $\overline{S}^c$ by k, A_k and $\overline{S}_k^c$, respectively:

$$F_{\overline{S}_k^c}(K) = \max\left\{p \in A_k \mid \sum_{i=1}^{k} w_i \overline{F}_{X_i}^{-1}(p) \le K\right\}, \tag{176}$$

with the set A_k given by

$$A_k = \left\{\overline{F}_{X_i}(K_{i,j}) \mid i = 1,\ldots,k \text{ and } j = 0,1,\ldots,m_i\right\} \setminus \{0\}. \tag{177}$$

From Theorem 7, we find that

$$\overline{C}_k^c[K] = \sum_{i=1}^{k} w_i \overline{C}_i\left[\overline{F}_{X_i}^{-1(\alpha_K^{(k)})}\left(F_{\overline{S}_k^c}(K)\right)\right], \tag{178}$$

$$\overline{P}_k^c[K] = \sum_{i=1}^{k} w_i \overline{P}_i\left[\overline{F}_{X_i}^{-1(\alpha_K^{(k)})}\left(F_{\overline{S}_k^c}(K)\right)\right], \tag{179}$$

with $\alpha_K^{(k)}$ determined from

$$\overline{F}_{\overline{S}_k^c}^{-1(\alpha_K^{(k)})}\left(F_{\overline{S}_k^c}(K)\right) = K. \tag{180}$$

Theorem 8 can be applied to determine alternative expressions for $\overline{C}_k^c[K]$ and $\overline{P}_k^c[K]$.

In the following theorem we prove that calculating the upper bounds $\overline{C}^c[K]$ and $\overline{P}^c[K]$ for the index call and put option prices reduces to the calculation of $\overline{C}_k^c[K]$ and $\overline{P}_k^c[K]$, provided the 'maximal' values K_{i,m_i+1}, $i = k+1,\ldots,n$, fulfill the following condition:

$$\max_{i \le k} \overline{F}_{X_i}(K_{i,m_i}) < \min_{i > k} \overline{F}_{X_i}(0). \tag{181}$$

From (172) it is clear that condition (181) is fulfilled provided the values $K_{i,1}$ of the last $(n-k)$ stocks are chosen sufficiently large.

Theorem 14 *Suppose that $m_i > 0$ for stocks $i = 1,2,\ldots,k$, while $m_i = 0$ for $i = k+1,\ldots,n$. Assume that the K_{i,m_i+1}, $i = 1,\ldots,n$, are chosen sufficiently large in the sense that (181) holds. In this case, we have that for any $K \in \left(\overline{F}_{\overline{S}^c}^{-1+}(0), \sum_{i=1}^{k} w_i K_{i,m_i+1}\right)$ the upper bounds $\overline{C}^c[K]$ and $\overline{P}^c[K]$ for the index option prices $C[K]$ and $P[K]$ are given by*

$$\overline{C}^c[K] = \overline{C}_k^c[K] + \sum_{i=k+1}^{n} w_i C_i[0] \tag{182}$$

and

$$\overline{P}^c[K] = \overline{P}_k^c[K], \tag{183}$$

respectively.

Proof. Let $K \in \left(F_{\overline{S}^c}^{-1+}(0), \sum_{i=1}^{k} w_i K_{i,m_i+1}\right)$. Taking into account that $F_{\overline{S}_k^c}^{-1+}(0) \leq F_{\overline{S}^c}^{-1+}(0)$ and $F_{\overline{S}_k^c}^{-1}(1) = \sum_{i=1}^{k} w_i K_{i,m_i+1}$, we have that $K \in \left(F_{\overline{S}_k^c}^{-1+}(0), F_{\overline{S}_k^c}^{-1}(1)\right)$.
From assumption (181) we find that

$$0 < p < \overline{F}_{X_i}(0), \qquad p \in A_k \text{ and } i > k. \tag{184}$$

From (172) it follows then that

$$\overline{F}_{X_i}^{-1}(p) = 0, \qquad p \in A_k \text{ and } i > k. \tag{185}$$

Notice that we find from (145), that $F_{\overline{S}^c}(K) \leq F_{\overline{S}_k^c}(K)$ always holds. On the other hand, (185) leads to

$$\begin{aligned} F_{\overline{S}_k^c}(K) &= \max\left\{p \in A_k \mid \sum_{i=1}^{k} w_i \overline{F}_{X_i}^{-1}(p) \leq K\right\} \\ &= \max\left\{p \in A_k \mid \sum_{i=1}^{n} w_i \overline{F}_{X_i}^{-1}(p) \leq K\right\} \\ &\leq \max\left\{p \in A \mid \sum_{i=1}^{n} w_i \overline{F}_{X_i}^{-1}(p) \leq K\right\} \\ &= F_{\overline{S}^c}(K). \end{aligned}$$

We can conclude that

$$F_{\overline{S}^c}(K) = F_{\overline{S}_k^c}(K). \tag{186}$$

We have that $F_{\overline{S}^c}(K) \in A_k$. We find from (184) that

$$\overline{F}_{X_i}^{-1(\alpha_K)}\left(F_{\overline{S}^c}(K)\right) = 0, \qquad i > k,$$

from which it follows that

$$\sum_{i=1}^{n} F_{X_i}^{-1(\alpha_K)}\left(F_{\overline{S}^c}(K)\right) = \sum_{i=1}^{k} F_{X_i}^{-1(\alpha_K)}\left(F_{\overline{S}^c}(K)\right).$$

Taking into account (180), this equality also implies

$$\alpha_K = \alpha_K^{(k)}. \tag{187}$$

The equalities (186) and (187) lead to

$$\overline{F}_{X_i}^{-1(\alpha_K)}\left(F_{\overline{S}^c}(K)\right) = \overline{F}_{X_i}^{-1(\alpha_K^{(k)})}\left(F_{\overline{S}_k^c}(K)\right), \qquad i = 1, 2, \ldots, n.$$

Hence, from Theorem 7 and (178), we finally find that

$$\begin{aligned}\overline{C}^c[K] &= \sum_{i=1}^{n} w_i \overline{C}_i \left[\overline{F}_{X_i}^{-1(\alpha_K)}\left(F_{\overline{S}^c}(K)\right)\right] \\ &= \sum_{i=1}^{k} w_i \overline{C}_i \left[\overline{F}_{X_i}^{-1(\alpha_K^{(k)})}\left(F_{\overline{S}_k^c}(K)\right)\right] + \sum_{i=k+1}^{n} w_i \overline{C}_i[0] \\ &= \overline{C}_k^c[K] + \sum_{i=k+1}^{n} w_i C_i[0].\end{aligned}$$

Similarly, one can prove that $\overline{P}^c[K] = \overline{P}_k^c[K]$. ■

From the previous theorem, we can conclude that under the assumption (181), the calculation of the upper bounds $\overline{C}^c[K]$ and $\overline{P}^c[K]$ reduces to the calculation of $\overline{C}_k^c[K]$ and $\overline{P}_k^c[K]$, if $K \in \left(F_{\overline{S}^c}^{-1+}(0), \sum_{i=1}^{k} w_i K_{i,m_i+1}\right)$. Notice that the requirement $K < \sum_{i=1}^{k} w_i K_{i,m_i+1}$ will be met, provided the K_{i,m_i+1}, $i > k$, are chosen sufficiently large. Finally, notice that in case the last $(n-k)$ stocks pay no dividends, each $C_i[0]$, $i = k+1, \ldots, n$, in (182) is equal to the corresponding current stock price $X_i(0)$. On the other hand, in case of dividend-paying stocks, we have that $C_i[0] \leq X_i(0)$, $i = k+1, \ldots, n$, and replacing each $C_i[0]$ by $X_i(0)$ results in an upper bound for $\overline{C}^c[K]$ and hence also for the index option price $C[K]$.

6 Final remarks

In a *model-based* approach, index (and other exotic) call option prices are determined via simulation techniques or via an appropriate approximation technique. We refer to Deelstra et al. (2004), where comonotonic approximations are used to determine the price of an index option, given that the underlying stock prices are modelled by a multivariate Black-Scholes model. Another approach consists of determining upper bounds for European-type index option prices, which are only based on available market information, without assuming any particular model for the underlying stock prices. Such an approach is called *model-free*.

To the best of our knowledge, Simon et al. (2000) were the first to use the theory of comonotonicity to derive model-free upper bounds for Asian options. They showed that this upper bound can be expressed in terms of European options. Albrecher et al. (2005) show that this model-independent upper bound for an Asian option price corresponds to a static super-replicating strategy for this option. They explain how a long position in an Asian option can be hedged by shorting a portfolio of European call options on the underlying stocks.

Hobson et al. (2005) derive a model-independent upper bound for index options. They considered Lagrange optimization techniques to construct an upper bound for the index option price as well as the corresponding super-replicating strategy. These authors also presented a more realistic framework, where the set of traded European options is finite. They showed that their upper bound is the lowest upper bound for the price of the index option which is consistent with the observed prices of the traded European options on the individual stocks contained in the index.

Chen et al. (2008) unify the approaches of Simon et al. (2000) and Hobson et al. (2005) by determining upper bounds for a general class of exotic options (including Asian and index options), based on the theory on comonotonicity.

In the current paper, we have presented the above-mentioned results for index options in a broader context. Different from the existing literature, we do not only consider index calls but also index puts. Moreover, it is shown that treating the pricing of an index call option and an index put option in an integrated framework results in an efficient way to calculate both upper bounds. We have added several extensions to the existing literature. In particular, we have considered the situation where for some of the constituent stocks in the index there are no options available.

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