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Daniël Linders, Fan Yang

AFI_16108

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Daniël Linders[a]* Fan Yang[b]†

[a]KU Leuven, Leuven, Belgium

[b]Department of Statistics and Actuarial Science, University of Waterloo, Waterloo, ON N2L 3G1, Canada

Version: June 22, 2016

Abstract

We consider the problem of aggregating dependent risks in the presence of partial dependence information. More concretely, we assume that the risks involved belong to independent subgroups and the dependence structure within each group is unknown. We show that a sharp convex upper bound exists in this setting and that the constrained upper bound improves the existing, unconstrained, comonotonic upper bound in convex order. Moreover, we characterize the constrained upper bound in terms of the distribution of its sum. Numerical illustrations are provided to show the improvement of the new upper bound.

1 Introduction

In various actuarial and financial applications, one has to deal with sums of dependent random variables. For example, calculating the level of solvency capital requires the aggregation of the different risks that an insurance or financial company is facing. Another example involving aggregating dependent risks is the pricing of basket options and other multivariate derivative products. Determining basket option prices, solvency capital, etc. requires a model that jointly describes the risks involved. In this paper we assume that the marginal risks are known, but the dependence structure is unknown.

A simplifying approach to tackle the problems arising when aggregating risks, is to assume independence between the various risks. In this setting, efficient numerical procedures exist to determine the distribution of the aggregated risk. For example, one can

*Corresponding author: daniel.linders@kuleuven.be

†fan.yang@uwaterloo.ca

use the algorithm proposed in Panjer (1981) or use Fourier methods, as was proposed in Heckman & Meyers (1983). A discussion of these methods is given in Embrechts & Frei (2009). Although the independence assumption is tractable from a computational point of view, it is not a realistic choice; in most actuarial and financial problems there are dependencies between the risks involved which have to be accounted for. A popular approach to build in the dependencies between the risks is to use an appropriate copula function; see e.g. Nelsen (2006). The use (and abuse) of copulas for finance and actuarial science is very well and extensively discussed in Frees & Valdez (1998) and Mikosch (2006). Alternatively, one can directly model the dependence by a parametric model such as the multivariate Erlang distribution (Lee & Lin (2012)), multivariate Variance Gamma (Luciano & Schoutens (2006)), etc.

Reliable and efficient estimation procedures for one-dimensional risks are available, but the extension to the multivariate case is less straightforward, which leads to a high degree of model risk. In many financial and actuarial situation, special attention has to be paid to the risk of underestimating the extend to which the risks involved move together; see e.g. Denuit et al. (2008). Therefore, one can opt for a prudent approach and search for the worst case dependence structure in the set of multivariate risks with given marginal distributions. A measure for the degree of model risk can then be developed by quantifying the distance between the model and the worst case dependence scenario; see e.g. Dhaene et al. (2009), Dhaene, Linders, Schoutens & Vyncke (2012), Cont & Deguest (2012) and Linders et al. (2015).

The notion of comonotonicity is closely connected to the search for worst case dependence scenarios; see e.g. Hoeffding (1940) and Dhaene et al. (2002a). A given set of risks is comonotonic if they are all moving in unison. Assuming a comonotonic dependence structure among the marginal risks leads to a worst case scenario, i.e. and upper bound, in convex order; see Meilijson & Nádas (1979), Kaas et al. (2002) and Dhaene et al. (2006). Whereas convex upper bounds are studied already for a long time, the literature on convex lower bounds is less extensive. It turns out that finding lower bounds for convex risk measures is closely linked to defining extreme negative dependence structures. A possible notion of extreme negative dependence is complete/joint mixability. This extreme negative dependence structure was studied in Wang & Wang (2011) and Wang et al. (2013) for finding convex lower bounds; see also Embrechts et al. (2013) and Bernard et al. (2014). Another approach for finding convex lower bounds is to employ the theory of mutual exclusivity; see e.g. Dhaene & Denuit (1999), Cheung & Lo (2013) and Cheung & Lo (2014). The notion of tail-mutual exclusivity and the search for lower bounds for Tail Value-at-Risk was considered in Cheung, Denuit & Dhaene (2015).

Employing a worst case dependence scenario to determine solvency requirements or other actuarial quantities is a prudent strategy, but often leads to unacceptable high risk levels. Indeed, in the worst case scenario, the different risks are non-compensating (or comonotonic) and no diversification is taken into account. However, one can think of situations where information about pairwise correlations is available. For example, when historical data of the marginal risks is available, statistical tools can be employed to back out the marginal distributions and the correlation between the risks. Adding this information does not allow to unambiguously determine the joint distribution function (i.e.

the copula), but it may be used to find an improved (i.e. sharper), and more acceptable, bound for the aggregated risk. Determining bounds with partial dependence information was already considered in Bernard et al. (2015), where worst case bounds for the Value-at-Risk are derived in the constrained case , where partial dependence information is available through the variance of the sum. In Bignozzi et al. (2015), the authors show how the worst case bound can be improved by assuming positive dependence information, whereas the best case bound can be improved when assuming negative dependence information.

In this paper we search for an improved convex upper bound for a sum of dependent random variables, by adding information about the dependence between fixed group of random variables. In a first step, we use the results of Cheung & Vanduffel (2012) to prove that no convex maximal element exists in the class of distributions with given marginals and given correlation matrix. We then restate our problem by relaxing the assumption that the full correlation matrix is given. More precisely, we assume that the different risks can be grouped in different subgroups. We assume independence between the different subgroups, but the dependence within each group is unknown. Such a situation can occur when aggregating risks over different lines of businesses. Assuming independence between the different lines of businesses may be justified because they operate in different segments or different geographical areas. The risks within the different lines of business are, of course, dependent. We show that in this setting, a convex maximal element exists if there is at least one group containing more than one element. Moreover, if there are at least 2 groups, this new convex bound improves the comonotonic bound. A numerical illustration showing the improvement of the upper bound one can obtain by including the dependence information is provided.

2 Stochastic orders, comonotonicity and risk measures

Consider the random vector $\underline{X} = (X_1, X_2, \ldots, X_n)$ and denote the sum of its components by S_X, so we have that $S_X = X_1 + X_2 + \ldots + X_n$. The cdf of X_i is denoted by F_{X_i}. Throughout this paper we assume that all random variables have finite mean and are defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We assume that all expectations we encounter in this paper are well-defined and finite.

2.1 Convex and supermodular orders

We first introduce the convex order.

Definition 1 (Convex order) *Consider two r.v.'s X and Y. Then X is said to precede Y in the* convex order sense*, notation $X \preceq_{cx} Y$, if*

$$X \preceq_{cx} Y \Leftrightarrow \begin{cases} \mathbb{E}[X] = \mathbb{E}[Y], \\ \mathbb{E}\left[(X-K)_+\right] \ \leq \mathbb{E}\left[(Y-K)_+\right], \text{ for all } K \in \mathbb{R}. \end{cases} \tag{1}$$

The convex order can equivalently be characterized as follows:

$$X \preceq_{cx} Y \Leftrightarrow \mathbb{E}\left[f\left(X\right)\right] \leq \mathbb{E}\left[f\left(Y\right)\right], \text{ for any convex function } f.$$

We immediately find that $X \preceq_{cx} Y \Rightarrow \mathrm{Var}[X] \leq \mathrm{Var}[Y]$, which shows that the random variable Y is 'more variable' than the random variable X. Moreover, one has that if $X \preceq_{cx} Y$, the following equivalence relation holds

$$\mathrm{Var}\left[X\right] = \mathrm{Var}\left[Y\right] \Leftrightarrow X \stackrel{\mathrm{d}}{=} Y.$$

A proof of this relation is given in Cheung & Vanduffel (2012) and Dhaene, Linders, Schoutens & Vyncke (2012). A generalization of this result can be found in Cheung, Dhaene, Kukush & Linders (2015). For more details on convex order, we refer to Shaked & Shanthikumar (1997) and Denuit et al. (2005).

Before introducing the supermodular order, we have to define supemodular functions. For any arbitrary function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, real-valued n - vector $\underline{x} = (x_1, x_2 \ldots, x_n)$, integer $i \in \{1, 2, \ldots, n\}$ and positive real number ε, the notation $\Delta_i^{\varepsilon} f\left(\underline{x}\right)$ is defined by

$$\Delta_i^{\varepsilon} f\left(\underline{x}\right) = f\left(x_1, x_2 \ldots, x_i + \varepsilon, x_{i+1}, \ldots, x_n\right) - f\left(x_1, x_2, \ldots, x_n\right).$$

Definition 2 (Supermodular function) *A function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is said to be supermodular if*

$$\Delta_j^{\delta} \Delta_i^{\varepsilon} f\left(\underline{x}\right) \geq 0$$

holds for every $\underline{x} \in \mathbb{R}^n$, $1 \leq i < j \leq n$ and all $\delta, \varepsilon > 0$.

We are now ready to define the supermodular order.

Definition 3 (Supermodular order) *Consider two random vectors $\underline{X}$ and $\underline{Y}$. Then $\underline{X}$ is said to be smaller in the* supermodular order *than $\underline{Y}$, notation $\underline{X} \preceq_{sm} \underline{Y}$, if*

$$\mathbb{E}\left[f\left(\underline{X}\right)\right] \leq \mathbb{E}\left[f\left(\underline{Y}\right)\right]$$

holds for all supermodular functions $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ for which the expectations exist.

If $\underline{X} \preceq_{sm} \underline{Y}$ then the random vectors $\underline{X}$ and $\underline{Y}$ have the same marginal distributions, i.e. $X_i = Y_i$, for $i = 1, 2, \ldots, n$, but the components of $\underline{Y}$ exhibit of stronger positive dependence than the components in $\underline{X}$. Indeed, we have that

$$\underline{X} \preceq_{sm} \underline{Y} \Rightarrow \begin{cases} \mathbb{P}\left[\underline{X} > \underline{x}\right] \leq \mathbb{P}\left[\underline{Y} > \underline{x}\right], & \text{for all } \underline{x} \in \mathbb{R}^n, \\ \mathbb{P}\left[\underline{X} \leq \underline{x}\right] \leq \mathbb{P}\left[\underline{Y} \leq \underline{x}\right], & \text{for all } \underline{x} \in \mathbb{R}^n. \end{cases}$$

Supermodular order also implies convex order of the sums of the respective components:

$$\underline{X} \preceq_{sm} \underline{Y} \Rightarrow S_X \preceq_{cx} S_Y, \tag{2}$$

see e.g. Proposition 6.3.9 in Denuit et al. (2005).

We consider the following lemma which shows that supermodular order is closed under conjunction; see also Theorem 9.A.9 in Shaked & Shanthikumar (1997).

Lemma 1 *Consider the $n-$dimensional vectors $\underline{X} = (\underline{X}_1, \underline{X}_2, \ldots, \underline{X}_k)$ and $\underline{Y} = (\underline{Y}_1, \underline{Y}_2, \ldots, \underline{Y}_k)$, where the subvectors $\underline{X}_l$ and $\underline{Y}_l$ are k_l dimensional. Moreover,*

$$\underline{X}_i \perp \underline{X}_j \text{ and } \underline{Y}_i \perp \underline{Y}_j \text{ for } i \neq j, \tag{3}$$

and

$$\underline{X}_i \preceq_{sm} \underline{Y}_i \text{ for } i = 1, 2, \ldots, k.$$

Then we have that

$$\underline{X} \preceq_{sm} \underline{Y}.$$

2.2 A convex maximal element in the Fréchet class

The random vector $(X_1, \ldots, X_n)$ is said to be *comonotonic* if

$$(X_1, \ldots, X_n) \stackrel{\mathrm{d}}{=} \left(F_{X_1}^{-1}(U), \ldots, F_{X_n}^{-1}(U)\right), \tag{4}$$

where U is a uniform $(0,1)$ r.v. The cdf of a comonotonic random vector $(X_1, \ldots, X_n)$ is given in terms of its marginal cdf's:

$$F_{(X_1,\ldots,X_n)}(x_1, x_2, \ldots, x_n) = \min\left\{F_{X_1}(x_1), F_{X_2}(x_2), \ldots, F_{X_n}(x_n)\right\}.$$

Consider a random vector $\underline{X}$, not necessarily comonotonic. We introduce the notation $\underline{X}^c = \left(F_{X_1}^{-1}(U), \ldots, F_{X_n}^{-1}(U)\right)$. We call $\underline{X}^c$ the comonotonic modification of the vector $\underline{X}$. The sum of the components of $\underline{X}^c$ is denoted by S^c. For an extensive overview of the theory of comonotonicity we refer to Dhaene et al. (2002a). An updated overview of applications of comonotonicity can be found in Deelstra et al. (2011).

Fix the random vector $\underline{X} = (X_1, X_2, \ldots, X_n)$ and the sum S_X. In order to determine actuarial and financial quantities for S_X, one has to model the marginals X_i, $i = 1, 2, \ldots, n$ and the dependence structure between the components. We assume that the marginal cdf F_{X_i} of X_i is known for each $i = 1, 2, \ldots, n$, but the dependence structure is unknown. The set $\mathcal{R}(\underline{X})$ of all random vectors which are consistent with the available marginal information is called the Fréchet class and is defined as

$$\mathcal{R}(\underline{X}) = \left\{(Y_1, Y_2, \ldots, Y_n) \mid F_{Y_i} \equiv F_{X_i},\ i = 1, 2, \ldots, n\right\}. \tag{5}$$

The convex maximal element in the set $\mathcal{R}(\underline{X})$ is the comonotonic vector; see e.g. Kaas et al. (2000). Indeed, we have that $\left(F_{X_1}^{-1}(U), \ldots, F_{X_n}^{-1}(U)\right) \in \mathcal{R}(\underline{X})$ and the following implication holds:

$$\underline{Y} \in \mathcal{R}(\underline{X}) \Longrightarrow Y_1 + Y_2 + \ldots + Y_n \preceq_{cx} F_{X_1}^{-1}(U) + F_{X_2}^{-1}(U) + \ldots + F_{X_n}^{-1}(U). \tag{6}$$

The converse relation was first proven in Cheung (2008) and generalized in Cheung (2010) and Cheung, Dhaene, Kukush & Linders (2015). We can thus state the following theorem.

Theorem 4 *Fix a random vector $\underline{Y}^* \in \mathcal{R}(\underline{X})$. Then, the following statements are equivalent*

1. *For any* $\underline{Y} \in \mathcal{R}(\underline{X})$ *, it holds that* $\underline{Y} \preceq_{sm} \underline{Y}^*$.
2. *For any* $\underline{Y} \in \mathcal{R}(\underline{X})$ *, it holds that* $Y_1 + Y_2 + \ldots + Y_n \preceq_{cx} Y_1^* + Y_2^* + \ldots + Y_n^*$.
3. $\underline{Y}^*$ *is comonotonic.*
4. $Var[Y_1^* + Y_2^* + \ldots + Y_n^*] = Var\left[F_{X_1}^{-1}(U) + F_{X_2}^{-1}(U) + \ldots + F_{X_n}^{-1}(U)\right]$ *, provided the variances are well-defined and finite.*

The previous theorem shows that a comonotonic random vector can be characterized by the distribution of its sum. Moreover, this comonotonic vector can be captured by a single real number, representing the variance of the sum.

2.3 Distortion risk measures and comonotonicity

Consider the set Γ containing real-valued random variables. We assume that $X_1, X_2 \in \Gamma$ implies that $X_1 + X_2 \in \Gamma$. Moreover, it is assumed that the set Γ satisfies the following conditions: if $X \in \Gamma$ and $a > 0$ and $b \in \mathbb{R}$, then $aX + b \in \Gamma$. The function $\rho : \Gamma \to \mathbb{R}$ attaches the real-number $\rho[X]$ to any random variable X in Γ and is called a *risk measure*. In the sequel we will always assume that the set Γ is taken 'as broad as possible' and all random variables we encounter belong to this set Γ.

Two random variables X and Y are said to be ordered in the stop-loss order, $X \preceq_{sl} Y$, if $\mathbb{E}[(X-K)_+] \le \mathbb{E}[(Y-K)_+]$, for all $K \in \mathbb{R}$. A risk measure ρ is said to preserve stop-loss order if it satisfies the following condition:

$$X \preceq_{sl} Y \Rightarrow \rho[X] \le \rho[Y].$$

For a risk measure ρ which preserves stop-loss order, we find that for any $\underline{Y} \in \mathcal{R}(\underline{X})$:

$$\rho\left[\sum_{i=1}^{n} X_i\right] \le \rho\left[\sum_{i=1}^{n} F_{X_i}^{-1}(U)\right].$$

This inequality indicates that choosing the comonotonic distribution is a prudent approach, in the sense that it will lead to a maximal risk level when employing the risk measure ρ for quantifying the risk.

A distortion function is defined as a non-decreasing function $g : [0,1] \to [0,1]$ such that $g(0) = 0$ and $g(1) = 1$. For any r.v. X, the distorted expectation associated with the distortion function g, notation $\rho_g[X]$, is defined by

$$\rho_g[X] = -\int_{-\infty}^{0} \left[1 - g\left(\overline{F}_X(x)\right)\right] \mathrm{d}x + \int_{0}^{+\infty} g\left(\overline{F}_X(x)\right) \mathrm{d}x, \tag{7}$$

provided at least one of the two integrals in (7) is finite. The functional ρ_g is called the *distortion risk measure*. We always assume that $\rho_g[X]$ is finite. For a detailed description of distorted expectations and distortion risk measures we refer to: Wang (1996), Gzyland & Mayoral (2006), Goovaerts et al. (2012).

It was proven in Wirch & Hardy (2000) that the risk measure ρ_g with a concave distortion function g is subadditive. Moreover, a distortion risk measure is always additive for comonotonic risks. A proof of this property for concave distortion functions is provided in Wang (1996), whereas a proof for the general situation is given in Dhaene, Kukush, Linders & Tang (2012). Finally, a concave distortion function leads to a distortion risk measure ρ_g which preserves stop-loss order. Hence, we can write the following for a concave distortion risk measure ρ_g :

$$\rho_g\left[\sum_{i=1}^{n} X_i\right] \leq \rho_g\left[S^c\right] = \sum_{i=1}^{n}\rho_g\left[X_i^c\right]. \tag{8}$$

A review on risk measures and comonotonicity can be found in Dhaene et al. (2006).

The class of distortion risk measures contains many important and widely used risk measures. The Value-at-Risk (VaR) of a r.v. X at level $p \in (0,1)$ is defined as

$$\mathrm{VaR}_p(X) = \inf\{x : F(x) \geq 1-p\}.$$

It is easy to verify that VaR is a distortion risk measure by choosing the distortion function to be

$$g_1(x) = 1_{\{x>1-p\}}, \qquad x \in [0,1].$$

Apparently, g_1 is not concave and hence, VaR is not subadditive; see e.g. Dowd & Blake (2006). Another important distortion risk measure is the Tail Value-at-Risk (TVaR). The TVaR of a r.v. X at level $p \in (0,1)$ is defined as

$$\mathrm{TVaR}_p(X) = \frac{1}{1-p}\int_p^1 \mathrm{VaR}_q(X)\mathrm{d}q.$$

It corresponds to a distortion risk measure with distortion function

$$g_2(x) = \min\left\{\frac{q}{1-p}, 1\right\}, \qquad x \in [0,1].$$

Since g_2 is concave, TVaR is a subadditive distortion risk measure. Lastly, the Wang transform (WT) risk measure is defined as

$$WT[X] = \rho_{g_3}[X],$$

where the distortion function is

$$g_3(x) = \Phi\left(\Phi^{-1}(x) + \Phi^{-1}(p)\right), \qquad x \in [0,1].$$

Since g_3 is a strictly concave distortion function, WT is also subadditive. Readers interested in more details about these risk measures are referred to Dhaene et al. (2006).

3 Convex upper bounds with dependence information

3.1 A useful lemma

We first consider the following lemma, which was already proven in Denuit et al. (2005) (Proposition 3.4.25). This lemma can be considered as the one-dimensional version of Lemma 1.

Lemma 2 *Consider the independent random vectors* $(X_1, X_2, \ldots, X_n)$ *and* $(Y_1, Y_2, \ldots, Y_n)$. *If we have that* $X_i \preceq_{cx} Y_i$, *for* $i = 1, 2, \ldots, n$, *then*

$$X_1 + X_2 + \ldots + X_n \preceq_{cx} Y_1 + Y_2 + \ldots + Y_n.$$

The previous lemma shows that under the appropriate conditions, aggregating independent risks X_i leads to a sum which is bounded from above in the convex order. The following lemma shows that if this upper bound is reached, the two vectors have to be equal in distribution. In this situation, the random vector can be characterized using the distribution of the sum. The proof is mainly based on the proof of Lemma 2. The following lemma plays an important role when proving our main result.

Lemma 3 *Consider the* $n-$*dimensional independent random vectors* $\underline{X}$ *and* $\underline{Y}$. *If we have that*

$$X_i \preceq_{cx} Y_i \text{ for } i = 1, 2, \ldots, n, \tag{9}$$

then:

$$S_X \overset{d}{=} S_Y \Leftrightarrow \underline{X} \overset{d}{=} \underline{Y}. \tag{10}$$

Proof. The proof of the $\Leftarrow$ statement is trivial.

We prove the $\Rightarrow$ statement by induction. If $n = 1$, the proof is direct. Assume that $n = 2$, then we have that

$$X_1 \perp X_2 \text{ and } Y_1 \perp Y_2,$$

and (10) can be written as

$$X_1 + X_2 \overset{\mathrm{d}}{=} Y_1 + Y_2 \Leftrightarrow (X_1, X_2) \overset{\mathrm{d}}{=} (Y_1, Y_2).$$

Because it is given that the copula of both $\underline{X}$ and $\underline{Y}$ is the independent copula, it suffices to prove that

$$X_i \overset{\mathrm{d}}{=} Y_i, \text{ for } i = 1, 2. \tag{11}$$

The conditions $X_i \preceq_{cx} Y_i$ can be written as

$$\mathbb{E}\left[(X_i - K)_+\right] \leq \mathbb{E}\left[(Y_i - K)_+\right], \text{ for all } K \in \mathbb{R}. \tag{12}$$

Proving (11) is equivalent with proving that (12) can only hold with equality. We prove (11) by contradiction. Denote the set $\{K : E[(X_i - K)_+] < E[(Y_i - K)_+]\}$ by A_i for $i = 1, 2$. Since the function $f_i(K) = E[(Y_i - K)_+] - E[(X_i - K)_+]$ is a right continuous function, we have either $\mu(A_i) > 0$ or $A_i = \emptyset$, where μ is the Lebesgue measure on $\mathbb{R}$. By using the tower property together with the independence between X_1 and X_2, the stop-loss premium $\mathbb{E}\left[(X_1 + X_2 - K)_+\right]$ can be written as follows

$$\begin{aligned}
\mathbb{E}\left[(X_1 + X_2 - K)_+\right] &= \mathbb{E}\left[\left[\mathbb{E}\left(X_1 + X_2 - K\right)_+ \mid X_1\right]\right] \\
&= \int_{\mathbb{R}} \mathbb{E}\left[(X_2 - (K - x))_+\right] \mathrm{d}F_{X_1}(x) \\
&= \int_{\mathbb{R}\setminus A_2} \mathbb{E}\left[(X_2 - (K - x))_+\right] \mathrm{d}F_{X_1}(x) + \int_{A_2} \mathbb{E}\left[(X_2 - (K - x))_+\right] \mathrm{d}F_{X_1}(x) \\
&< \int_{\mathbb{R}} \mathbb{E}\left[(Y_2 - (K - x))_+\right] \mathrm{d}F_{X_1}(x) \\
&= \mathbb{E}\left[(X_1 + Y_2 - K)_+\right].
\end{aligned}$$

The stop-loss premium $\mathbb{E}\left[(X_1 + Y_2 - K)_+\right]$ can be rewritten by employing the independence between X_1 and Y_2 :

$$\begin{aligned}
\mathbb{E}\left[(X_1 + Y_2 - K)_+\right] &= \int_{\mathbb{R}} \mathbb{E}\left[(X_1 - (K - y))_+\right] \mathrm{d}F_{Y_2}(y) \\
&< \int_{\mathbb{R}} \mathbb{E}\left[(Y_1 - (K - y))_+\right] \mathrm{d}F_{Y_2}(y) \\
&= \mathbb{E}\left[(Y_1 + Y_2 - K)_+\right].
\end{aligned}$$

We conclude that

$$\mathbb{E}\left[(X_1 + X_2 - K)_+\right] < \mathbb{E}\left[(X_1 + Y_2 - K)_+\right] < \mathbb{E}\left[(Y_1 + Y_2 - K)_+\right],$$

which contradicts $X_1 + X_2 \stackrel{\mathrm{d}}{=} Y_1 + Y_2$. We then find $E[(X_1 + X_2 - K)_+] = E[(Y_1 + Y_2 - K)_+]$ holds if and only if $A_i = \emptyset$ for $i = 1, 2$, whereas $A_i = \emptyset$ is equivalent to that $E[(Y_i - K)_+] = E[(X_i - K)_+]$ for $i = 1, 2$.

Assume now that the statement holds for $n - 1$. Note that

$$\sum_{i=1}^{n-1} X_i \perp X_n \text{ and } \sum_{i=1}^{n-1} Y_i \perp Y_n.$$

Furthermore, because (9) holds, it follows from Lemma 2 that

$$\sum_{i=1}^{n-1} X_i \preceq_{cx} \sum_{i=1}^{n-1} Y_i \text{ and } X_n \preceq_{cx} Y_n.$$

We can then use the 2-dimensional version of the theorem to show that:

$$\begin{gathered}
X_1 + X_2 + \ldots + X_{n-1} + X_n \stackrel{\mathrm{d}}{=} Y_1 + Y_2 + \ldots + Y_{n-1} + Y_n \\
\Updownarrow \\
\left(\sum_{i=1}^{n-1} X_i, X_n\right) \stackrel{\mathrm{d}}{=} \left(\sum_{i=1}^{n-1} Y_i, Y_n\right).
\end{gathered}$$

Hence we find that $\sum_{i=1}^{n-1} X_i \stackrel{\mathrm{d}}{=} \sum_{i=1}^{n-1} Y_i$, which is equivalent with $(X_1, X_2, \ldots, X_{n-1}) \stackrel{\mathrm{d}}{=} (Y_1, Y_2, \ldots, Y_{n-1})$. This then ends the proof. ■

Lemma 3 extends Lemma 2. Whereas Lemma 2 provides a convex upper bound for the sum S_X, Lemma 3 shows that there is only one possible choice for the marginals X_i of the vector $\underline{X}$ in order for S_X to reach its convex upper bound S_Y. Indeed, in order to reach the upper bound, each marginal distribution X_i should be set equal to its respective convex upper bound Y_i.

3.2 Convex upper bounds with known correlation matrix

Assume that all the pairwise correlations are given. We use the following notation:

$$\operatorname{Corr}[X_i, X_j] = \rho_{i,j}, \text{ for } i, j = 1, 2, \ldots, n,$$

where $\rho_{i,i} = 1$. The Fréchet space $\mathcal{R}(\underline{X})$, defined in (5), consists of all random vectors having the same marginals, but different dependence structures. If we have also information about the pairwise correlations, the new class $\mathcal{C}(\underline{X})$ of admissible random vectors is given by

$$\mathcal{C}(\underline{X}) = \{(Y_1, Y_2, \ldots, Y_n) \in \mathcal{R}(\underline{X}) \mid \operatorname{Corr}[Y_i, Y_j] = \rho_{i,j}\ i, j = 1, 2, \ldots, n\}.$$

Note that $\mathcal{C}(\underline{X}) \subset \mathcal{R}(\underline{X})$, which shows that adding dependence information reduces the set of feasible random vectors.We assume that the correlation matrix is chosen such that the cardinality of the set $\mathcal{C}(\underline{X})$ is at least 1. However, the set $\mathcal{C}(\underline{X})$ may contain more than one element, indicating that the pairwise correlations do not unambiguously specify the copula. One may then hope that a convex maximal element can be found in the class $\mathcal{C}(\underline{X})$. It was already proven in Cheung & Vanduffel (2012) that no convex maximal element exists in the class of random vectors with fixed marginals and finite variance. In a similar way, we can then prove the following result.

Theorem 5 *There does not exist a convex maximal element in the class $\mathcal{C}(\underline{X})$.*

Proof. If all pairwise correlation are known, also the variance $\operatorname{Var}[\sum_{i=1}^{n} X_i]$, which we also denote by c, is known. Define the set $\mathcal{A}$ as

$$\mathcal{A} = \left\{(Y_1, Y_2, \ldots, Y_n) \in \mathcal{R}(\underline{X}) \mid \operatorname{Var}\left[\sum_{i=1}^{n} Y_i\right] = c,\ i = 1, 2, \ldots, n\right\}. \tag{13}$$

It was proven in Theorem 7 in Cheung & Vanduffel (2012) that no convex maximal element can exist in the set $\mathcal{A}$. Noting that $\mathcal{C}(\underline{X}) \subset \mathcal{A}$ then ends the proof. ■

Note that in a similar way, we can also show that a convex minimal element in $\mathcal{C}(\underline{X})$ will not exist. It turns out that imposing constraints on all the pairwise correlations makes the class $\mathcal{C}(\underline{X})$ to narrow, in the sense that no convex order between the elements in this class can exist. Therefore, we will consider the following, related problem. We assume that the risks $X_1, X_2, \ldots, X_n$ can be divided in different groups. Random variables belonging to different groups are assumed to be independent. However, the dependence structure between the r.v.'s composing a group is unknown. We will search for a convex maximal element in this setting.

3.3 A convex maximal element when the marginals and some correlations are fixed

Assume that the risks $X_1, X_2, \ldots, X_n$ can be grouped in k disjoint sets. Define the indices $i_1, i_2, \ldots, i_k$ such that $0 < i_1 < i_2 < \ldots < i_k = n$. Without loss of generality, we can assume that the r.v.'s $X_1, X_2, \ldots, X_{i_1}$ belong to group 1, the r.v.'s $X_{i_1+1}, X_{i_1+2}, \ldots, X_{i_2}$ are in group 2. Continuing like this, we find that the $k-$th group contains the r.v.'s $X_{i_{k-1}+1}, X_{i_{k-1}+2}, \ldots, X_{i_k}$. Denote the set containing the indices of group l, $l = 1, 2, \ldots, k$ by I_l, so

$$I_l = \{i_{l-1} + 1, i_{l-1} + 2 \ldots, i_l\},$$

with the convention that $i_0 = 0$. So for any $i \in I_l$, the r.v. X_i belongs to group l. We assume that the different groups are independent, i.e.

$$X_i \perp X_j, \text{ for } i \in I_l, \ j \in I_m \text{ and } l \neq m.$$

This also means that

$$\text{Corr}\left[X_i, X_j\right] = 0, \text{ for } i \in I_l, \ j \in I_m \text{ and } l \neq m.$$

Note, however, that the converse implication does not hold in general. Indeed, one can construct dependent random variables with pairwise correlations equal to zero.

The correlation $\text{Corr}[X_i, X_j]$ between two r.v.'s X_i and X_j belonging to the same group is not specified. We are looking for a convex maximal element in the set $\mathcal{I}_k(\underline{X})$:

$$\mathcal{I}_k(\underline{X}) = \left\{ (Y_1, Y_2, \ldots, Y_n) \in \mathcal{R}(\underline{X}) \left| \begin{array}{l} Y_i \perp Y_j, \text{ for } l, m = 1, 2, \ldots k, \\ l \neq m \text{ and } i \in I_l, \ j \in I_m \end{array} \right. \right\}.$$

Note that in case $k = 1$, all marginals belong to the same group and no independence information is used. In this case, $\mathcal{I}_k(\underline{X})$ corresponds with the Fréchet class defined in (5), i.e. $\mathcal{I}_k(\underline{X}) = \mathcal{R}(\underline{X})$. In this case, a convex maximal element exists and it is given by the comonotonic vector; see Theorem 4. If $k = n$, each group consists of a single element. The independence assumption leads to full knowledge about the copula. In this particular case, the only random vector satisfying the available marginal and dependence information is the independent copy of the vector $\underline{X}$.

Lemma 4 *Consider the independent uniform r.v.'s $U_1, U_2, \ldots, U_k$. Define the subvectors $\underline{Z}_j$ as follows*

$$\underline{Z}_j = \left(F^{-1}_{X_{i_{j-1}+1}}(U_j), F^{-1}_{X_{i_{j-1}+2}}(U_j), \ldots, F^{-1}_{X_{i_j}}(U_j) \right), \ j = 1, 2, \ldots, k. \tag{14}$$

Then, the vector $\underline{Z} = (\underline{Z}_1, \underline{Z}_2, \ldots, \underline{Z}_k)$ belongs to the class $\mathcal{I}_k(\underline{X})$.

Proof. Remark that in case $k = n$, we find that $\underline{X}^{\perp} \stackrel{\mathrm{d}}{=} \underline{Z}$. In order to exclude this trivial case, we assume from now on that $k < n$. It is straightforward to see that $Z_i \stackrel{\mathrm{d}}{=} X_i$.

Furthermore, independence between U_l and U_m, for $l \neq m$ implies also that for $i \in I_l$, $j \in I$, $F_{X_i}^{-1}(U_l)$ and $F_{X_j}^{-1}(U_m)$ are independent. ■

The random vector $\underline{Z}$ defined in the previous lemma is a particular element of the set $\mathcal{I}_k(\underline{X})$. In the sequel of the paper, we denote the independent modification of $(X_1, X_2, \ldots, X_n)$ by $\underline{X}^{\perp}$ whereas its comonotonic modification is denoted by $\underline{X}^c$.

Lemma 5 *Consider the random vector $\underline{Z}$ defined by (14). We have that*

$$\underline{X}^{\perp} \preceq_{sm} \underline{Z} \preceq_{sm} \underline{X}^c.$$

Proof. The independent modification $\underline{X}^{\perp}$ of $\underline{X}$ can be expressed as follows:

$$\underline{X}^{\perp} = \left(\underline{X}_1^{\perp}, \underline{X}_2^{\perp}, \ldots, \underline{X}_k^{\perp}\right),$$

where $\underline{X}_l^{\perp} = \left(X_{i_{l-1}+1}^{\perp}, X_{i_{l-1}+2}^{\perp}, \ldots, X_{i_l}^{\perp}\right)$, for $l = 1, 2, \ldots, k$. Similarly, the comonotonic modification $\underline{X}^c$ of $\underline{X}$ can be expressed as

$$\underline{X}^c = \left(\underline{X}_1^c, \underline{X}_2^c, \ldots, \underline{X}_k^c\right),$$

where $\underline{X}_l^c = \left(X_{i_{l-1}+1}^c, X_{i_{l-1}+2}^c, \ldots, X_{i_l}^c\right)$ for $l = 1, 2, \ldots, k$. Because $\underline{Z} \in \mathcal{I}_k(\underline{X})$, we directly find that

$$\underline{Z} \preceq_{sm} \underline{X}^c.$$

Note that from (14) , we find that the components of $\underline{Z}_l$ are comonotonic and hence:

$$\underline{Z}_l \stackrel{\mathrm{d}}{=} \underline{X}_l^c, \text{ for } l = 1, 2, \ldots k.$$

Then we find that both $\underline{Z}_l$ and $\underline{X}_l^{\perp}$ have the same marginals but different dependence structures. More precisely:

$$\underline{X}_l^{\perp} \preceq_{sm} \underline{Z}_l \text{ for } l = 1, 2, \ldots k. \tag{15}$$

Moreover, the subvectors $\underline{X}_l^{\perp}$ and $\underline{Z}_l$ of the vector $\underline{X}^{\perp}$ and $\underline{Z}$, respectively, are independent:

$$\underline{X}_i^{\perp} \perp \underline{X}_j^{\perp} \text{ and } \underline{Z}_i \perp \underline{Z}_j \text{ for } i \neq j. \tag{16}$$

Combining (15) and (16) with Lemma 1 results in $\underline{X}^{\perp} \preceq_{sm} \underline{Z}$, which proves the result. ■

A direct consequence of Lemma 5 is the following ordering of the correlation parameters:

$$0 \leq \operatorname{Corr}\left[Z_i, Z_j\right] \leq \operatorname{Corr}\left[F_{X_i}^{-1}(U), F_{X_j}^{-1}(U)\right], \text{ for } i, j = 1, 2, \ldots, n.$$

Furthermore, if we denote the sum of the components of $\underline{X}^{\perp}, \underline{Z}$ and $\underline{X}^c$ by $S_{X^{\perp}}, S_Z$ and S_{X^c}, respectively, we have the following convex order relation:

$$S_{X^{\perp}} \preceq_{cx} S_Z \preceq_{cx} S_{X^c}.$$

In the following theorem we show that $\underline{Z}$ is the convex maximal element of the set $\mathcal{I}_k(\underline{X})$.

Theorem 6 *Consider the random vector $\underline{X}$. Fix a random vector $\underline{Y}^* \in \mathcal{I}_k(\underline{X})$ and denote the corresponding sum by S_{Y^*}. The following statements are equivalent*

1. *For any $\underline{Y} \in \mathcal{I}_k(\underline{X})$, we have that $\underline{Y} \preceq_{sm} \underline{Y}^*$.*
2. *For any $\underline{Y} \in \mathcal{I}_k(\underline{X})$, we have that $S_Y \preceq_{cx} S_{Y^*}$.*
3. *$\underline{Y}^* \overset{d}{=} \underline{Z}$, where $\underline{Z}$ is defined by (14).*

Proof. We start with a proof for (3) $\Rightarrow$(1). The random vector $\underline{Z}$ consists of k independent subvectors $\underline{Z}_j$ where each of these subvectors is comonotonic. Using the notation $\underline{Y}_j = \left(Y_{i_{j-1}+1}, Y_{i_{j-1}+2}, \ldots, Y_{i_j}\right)$, it follows that

$$\underline{Y}_j \preceq_{sm} \underline{Z}_j, \text{ for } j = 1, 2, \ldots, k.$$

Using independence of the subvectors $\underline{Y}_1, \underline{Y}_2, \ldots, \underline{Y}_k$ and $\underline{Z}_1, \underline{Z}_2, \ldots, \underline{Z}_k$ together with Lemma 1 proves the result.

The implication (1)$\Rightarrow$(2) is a direct consequence of the implication (2).

Finally, we prove (2)$\Rightarrow$(3): As $\underline{Z} \in \mathcal{I}_k(\underline{X})$, we have that $S_Z \preceq_{cx} S_{Y^*}$. In the previous part of the proof, we showed that the vector $\underline{Z}$ is maximal in convex order. Because $\underline{Y}^* \in \mathcal{I}_k(\underline{X})$, we find that also $S_{Y^*} \preceq_{cx} S_Z$ must hold. Combining these two convex order inequalities yields:

$$S_Z \overset{d}{=} S_{Y^*}.$$

Because both $\underline{Y}^*$ and $\underline{Z}$ belong to the set $\mathcal{I}_k(\underline{X})$, the random sums S_{Y^*} and S_Z can be decomposed in k independent subgroups:

$$\sum_{j=1}^{i_1} Y_j^* + \sum_{j=i_1+1}^{i_2} Y_j^* + \ldots + \sum_{j=i_{k-1}+1}^{i_k} Y_j^* \overset{d}{=} \sum_{j=1}^{i_1} F_{X_j}^{-1}(U_1) + \sum_{j=i_1+1}^{i_2} F_{X_j}^{-1}(U_2) + \ldots + \sum_{j=i_{k-1}+1}^{i_k} F_{X_j}^{-1}(U_k). \tag{17}$$

Take $\underline{Y} \in \mathcal{I}_k(\underline{X})$. For $l = 1, 2, \ldots, k$, we have that the random sums $\sum_{j=i_{l-1}+1}^{i_l} F_{X_j}^{-1}(U_l)$ and $\sum_{j=i_{l-1}+1}^{i_l} Y_l$ are independent and convex ordered, i.e.

$$\sum_{j=i_{l-1}+1}^{i_l} Y_j^* \preceq_{cx} \sum_{j=i_{l-1}+1}^{i_l} F_{X_j}^{-1}(U_l), \;\; l = 1, 2, \ldots, k. \tag{18}$$

Combining (17) and (18) together with Lemma 3 leads to:

$$\sum_{j=i_{l-1}+1}^{i_l} Y_j^* \overset{d}{=} \sum_{j=i_{l-1}+1}^{i_l} F_{X_j}^{-1}(U_l), l = 1, 2, \ldots, k. \tag{19}$$

Using Theorem 4, we then find that (19) implies

$$\left(Y_{i_{l-1}+1}^*, Y_{i_{l-1}+2}^*, \ldots, Y_{i_l}^*\right) \overset{\mathrm{d}}{=} \left(F_{X_{i_{l-1}+1}}^{-1}(U_l), F_{X_{i_{l-1}+2}}^{-1}(U_l), \ldots, F_{X_{i_l}}^{-1}(U_l)\right), \;\; l = 1, 2, \ldots, k.$$

By definition, the subvectors $\left(Y^*_{i_{l-1}+1}, Y^*_{i_{l-1}+2}, \ldots, Y^*_{i_l}\right)$, $l = 1, 2, \ldots, k$ are all independent. We can now conclude that $\underline{Y}^* \stackrel{\mathrm{d}}{=} \underline{Z}$. ■

Note that the information about the independence between groups of r.v.'s allows us to improve the comonotonic upper bound. Indeed, one has that

$$\begin{aligned}
\sum_{j=1}^{n} X_j &\preceq_{cx} \sum_{j=1}^{i_1} F_{X_j}^{-1}(U_1) + \sum_{j=i_1+1}^{i_2} F_{X_j}^{-1}(U_2) + \ldots + \sum_{j=i_{k-1}+1}^{i_k} F_{X_j}^{-1}(U_k) \\
&\preceq_{cx} \sum_{j=1}^{n} F_{X_j}^{-1}(U),
\end{aligned}$$

where U, $U_1, U_2, \ldots, U_k$ are independent uniform r.v.'s.

3.4 A simple proof for Theorem 6

In order to prove Theorem 6, it is only required that the marginal distributions of the vector $\underline{X}$ have finite mean. However, if we additionally assume that also the marginal variances are finite, we can give a more direct proof of this theorem. To be more precise, the most difficult step in the proof is the following:

$$S_{Y^*} \stackrel{\mathrm{d}}{=} S_Z \Longrightarrow \underline{Y}^* \stackrel{\mathrm{d}}{=} \underline{Z}, \tag{20}$$

where $\underline{Z}$ is defined by (14). We now provide an alternative proof for implication (20).

From $S_{Y^*} \stackrel{\mathrm{d}}{=} S_Z$, we find that

$$\mathrm{Var}\left[S_{Y^*}\right] = \mathrm{Var}\left[S_Z\right],$$

which can be rewritten as

$$\sum_{l=1}^{k} \left(\mathrm{Var}\left[\sum_{j=i_{l-1}+1}^{i_l} F_{X_j}^{-1}(U_l) \right] - \mathrm{Var}\left[\sum_{j=i_{l-1}+1}^{i_l} Y_j^* \right] \right) = 0. \tag{21}$$

From the convex order relation (18), we find

$$\mathrm{Var}\left[\sum_{j=i_{l-1}+1}^{i_l} F_{X_j}^{-1}(U_l) \right] - \mathrm{Var}\left[\sum_{j=i_l+1}^{i_l} Y_j^* \right] \geq 0, \text{ for } l = 1, 2, \ldots, k. \tag{22}$$

From inequality (22) we find that (21) can only hold if

$$\mathrm{Var}\left[\sum_{j=i_{l-1}+1}^{i_l} F_{X_j}^{-1}(U_l) \right] = \mathrm{Var}\left[\sum_{j=i_l+1}^{i_l} Y_j^* \right], \qquad l = 1, 2, \ldots, k, . \tag{23}$$

Note that $\left(F^{-1}_{X_{i_{l-1}+1}}\left(U_l\right), F^{-1}_{X_{i_{l-1}+2}}\left(U_l\right), \ldots, F^{-1}_{X_{i_l}}\left(U_l\right)\right)$ is the comonotonic modification of $\left(Y^*_{i_{l-1}+1}, Y^*_{i_{l-1}+2}, \ldots, Y^*_{i_l}\right)$. Using Theorem 4, we then find that (23) implies

$$\left(Y^*_{i_{l-1}+1}, Y^*_{i_{l-1}+2}, \ldots, Y^*_{i_l}\right) \stackrel{\mathrm{d}}{=} \left(F^{-1}_{X_{i_{l-1}+1}}\left(U_l\right), F^{-1}_{X_{i_{l-1}+2}}\left(U_l\right), \ldots, F^{-1}_{X_{i_l}}\left(U_l\right)\right), \ l = 1, 2, \ldots, k.$$

By definition, the subvectors $\left(Y^*_{i_{l-1}+1}, Y^*_{i_{l-1}+2}, \ldots, Y^*_{i_l}\right)$, $l = 1, 2, \ldots, k$ are all independent. We can now conclude that $\underline{Y}^* \stackrel{\mathrm{d}}{=} \underline{Z}$.

4 Distortion risk measures for sums of lognormals

In this section we make a particular distributional choice for the marginals. we consider the situation where the random variables $X_1, X_2, \ldots, X_n$ are lognormal distributed, i.e.:

$$X_i = \mathrm{e}^{Y_i}, \quad i = 1, 2, \ldots, n. \tag{24}$$

The random vector $(Y_1, Y_2, \ldots, Y_n)$ is multivariate normal with marginals given by

$$Y_i \stackrel{\mathrm{d}}{=} N\left(\mu_i, \sigma_i^2\right), \ i = 1, 2, \ldots, n$$

and correlation given by

$$\rho_{i,j} = \mathrm{Corr}\left[Y_i, Y_j\right], \ i \neq j.$$

The sum S is defined as

$$S_X = X_1 + X_2 + \ldots + X_n,$$

and is a sum of lognormal random variables. In this section we drop the subscript X if no confusion is possible and write S instead of S_X.

Although no explicit expression for the cdf F_S of S exists, the first three moments of S are given in closed form. Indeed, introduce the following notation

$$m_i = \mathbb{E}\left[S^i\right], \ \text{for } i = 1, 2, 3.$$

Then we have that

$$m_1 = \sum_{i=1}^{n} \mathbb{E}\left[X_i\right], \tag{25}$$

$$m_2 = \sum_{i=1}^{n}\sum_{j=1}^{n} \mathbb{E}\left[X_i\right]\mathbb{E}\left[X_j\right]\mathrm{e}^{\rho_{i,j}\sigma_i\sigma_j}, \tag{26}$$

$$m_3 = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} \mathbb{E}\left[X_i\right]\mathbb{E}\left[X_j\right]\mathbb{E}\left[X_k\right]\mathrm{e}^{\rho_{i,j}\sigma_i\sigma_j}\mathrm{e}^{\rho_{i,k}\sigma_i\sigma_k}\mathrm{e}^{\rho_{j,k}\sigma_j\sigma_k}, \tag{27}$$

and

$$\mathbb{E}\left[X_i\right] = \mathrm{e}^{\mu_i + \frac{\sigma_i^{é}}{2}},\ \text{for } i = 1, 2, \ldots, n.$$

This Gaussian copula model with lognormal marginals is popular in practice because of its mathematical tractability.

Together with the subadditivity property we find that the following inequalities hold for a distortion risk measure ρ_g with concave distortion function g :

$$\rho_g\left[S\right] \leq \rho_g\left[\sum_{i=1}^{n} Z_i\right] \leq \rho_g\left[S^c\right],$$

where S is the aggregated risk, S^c is the comonotonic sum and $\sum_{i=1}^{n} Z_i$ is the reduced convex bound defined in Theorem 6 . The quantity $\rho_g\left[S\right]$ can be interpreted as the aggregated risk, measured through the distortion risk measure ρ_g. If no information about the copula of $(X_1, X_2, \ldots, X_n)$ is available, $\rho_g\left[S\right]$ cannot be defined and the worst-case situation is characterized by the comonotonic sum S^c. The corresponding risk number is captured in the quantity $\rho_g\left[S^c\right]$. Adding information about the independent groups allows for a reduction of the worst-case situation, which is now given by $\rho_g\left[\sum_{i=1}^{n} Z_i\right]$.

In this section, we consider two possible choices for the distortion risk measures, TVaR and WT. For the lognormal marginals X_i, the TVaR and the Wang transform are given in closed form

$$\begin{aligned}
\mathrm{TVaR}_p\left[X_i\right] &= \mathrm{e}^{\mu_i + \frac{\sigma_i^2}{2}} \frac{\Phi\left(\sigma_i - \Phi^{-1}\left(p\right)\right)}{1-p}, \\
\mathrm{WT}_p\left[X_i\right] &= \mathrm{e}^{\mu_i + \frac{\sigma_i^2}{2} + \sigma_i \Phi^{-1}(p)}.
\end{aligned}$$

In case no dependence information is provided, the comonotonic upper bound S^c is the best convex bound we can use. Using the comonotonic additivity property (8), we find that

$$\mathrm{TVaR}_p\left[S^c\right] = \sum_{i=1}^{n} \mathrm{e}^{\mu_i + \frac{\sigma_i^2}{2}} \frac{\Phi\left(\sigma_i - \Phi^{-1}\left(p\right)\right)}{1-p}, \tag{28}$$

$$\mathrm{WT}_p\left[S^c\right] = \sum_{i=1}^{n} \mathrm{e}^{\mu_i + \frac{\sigma_i^2}{2} + \sigma_i \Phi^{-1}(p)}, \tag{29}$$

and

$$\mathrm{TVaR}_p\left[S\right] \leq \mathrm{TVaR}_p\left[S^c\right] \text{ and } \mathrm{WT}_p\left[S\right] \leq \mathrm{WT}_p\left[S^c\right].$$

4.1 The convex upper bound with dependence information

Consider the setup of Section 3, i.e., the random variables $X_1, X_2, \ldots, X_n$ are now assumed to be divided into k independent groups. The maximal convex upper bound including this additional dependence information is now given by:

$$S_Z^{(k)} = Z_1 + Z_2 + \ldots + Z_n,$$

where the random vector $\underline{Z}$ is defined in (14). Because X_i is lognormal distributed, also Z_i is lognormal distributed with the same parameters:

$$\log Z_i = N\left(\mu_i, \sigma_i^2\right), \quad i = 1, 2, \ldots, n.$$

However, the correlation matrix of the log-returns is different for $\underline{Z}$ than for $\underline{X}$. Indeed, we now have that

$$\text{Corr}\left[\log Z_i, \log Z_j\right] = \begin{cases} 1 & \text{There is a group } l \text{ such that } i, j \in l, \\ 0 & \text{else.} \end{cases}$$

If we denote this correlation matrix by R, we find

$$R = \begin{pmatrix} \overbrace{\begin{matrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{matrix}}^{1 \ldots i_1} & 0 & 0 & \cdots & 0 \\ 0 & \overbrace{\begin{matrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{matrix}}^{i_1+1 \ldots i_2} & 0 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \overbrace{\begin{matrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{matrix}}^{i_{k-1}+1 \ldots n} \end{pmatrix}$$

We determine the distortion risk measure ρ_g with concave g for the improved upper bound $S_Z^{(k)}$ giving an improvement of the worst-case aggregated risk:

$$\rho_g\left[S\right] \leq \rho_g\left[S_Z\right] < \rho_g\left[S^c\right].$$

If we can determine $\rho_g\left[X_i\right]$, the comonotonic upper bound $\rho_g\left[S^c\right]$ can be determined using the decomposition formula (8). In order to determine $\rho_g\left[S_Z^{(k)}\right]$, we have to determine a risk measure for a sum of lognormal random variables with known correlation matrix. This can be done using Monte-Carlo simulation or by using a numerical approximation. A possible approximation methodology yielding closed form expressions for $\rho_g\left[S_Z\right]$ is using a moment-matching method to approximate the sum S_Z by a shifted lognormal random variable having the same first three moments; see Brigo et al. (2004) and Linders & Schoutens (2015). Another methodology for deriving closed form approximations for $\rho_g\left[S_Z\right]$ is by employing the theory of comonotonicity; see Kaas et al. (2000) and Valdez et al. (2009).

4.2 Numerical illustration

Consider the situation where $n = 100$ and each X_i is lognormal distributed with $\mu_i = 0$ and $\sigma_i = 1$. In case we have k independent groups, we can determine the improved convex bound $\rho_g\left[S_Z^{(k)}\right]$. We consider the situations where $k = 2, 5, 10$ and 50. The improvement $\varepsilon_p^{(k)}$ is defined as

$$\varepsilon_p^{(k)} = \frac{\rho_g\left[S^c\right] - \rho_g\left[S_Z^{(k)}\right]}{\rho_g\left[S^c\right]}.$$

The improvement $\varepsilon_p^{(k)}$ indicates how much (expressed in percentages) the worst-case scenario improves when we add the information about the k independent groups.

The comonotonic and the improved upper bound are determined for TVaR and the Wang Transform. The comonotonic upper bounds $\mathrm{TVaR}_p\left[S^c\right]$ and $\mathrm{WT}_p\left[S^c\right]$ follows directly from expressions (28) and (29), respectively. The improved bounds $\mathrm{TVaR}_p\left[S_Z^{(k)}\right]$ and $\mathrm{WT}_p\left[S_Z^{(k)}\right]$ are determined using the moment-matching procedure. The results for the TVaR and the Wang transform are shown in Figures 1 and 2, respectively. The comonotonic bound (circles) corresponds with the worst-case risk measure if no dependence information is available. The bigger k, the more dependence information is revealed and hence, we observe that the worst-case risk measure with dependence information is decreasing in k. A similar conclusion can be drawn when we look at the improvements. The improvement is larger for larger values of k. From the results we can conclude that dividing the risks in only two independent groups already leads to an improvement of more than 10%.

Acknowledgement: *The authors acknowledge the support of the Society of Actuaries CKER (Risk Aggregation with Partial Dependence Information). Daniël Linders also acknowledges the financial support of the Onderzoeksfonds KU Leuven (GOA/13/002: Management of Financial and Actuarial Risks: Modeling, Regulation, Disclosure and Market Effects).*

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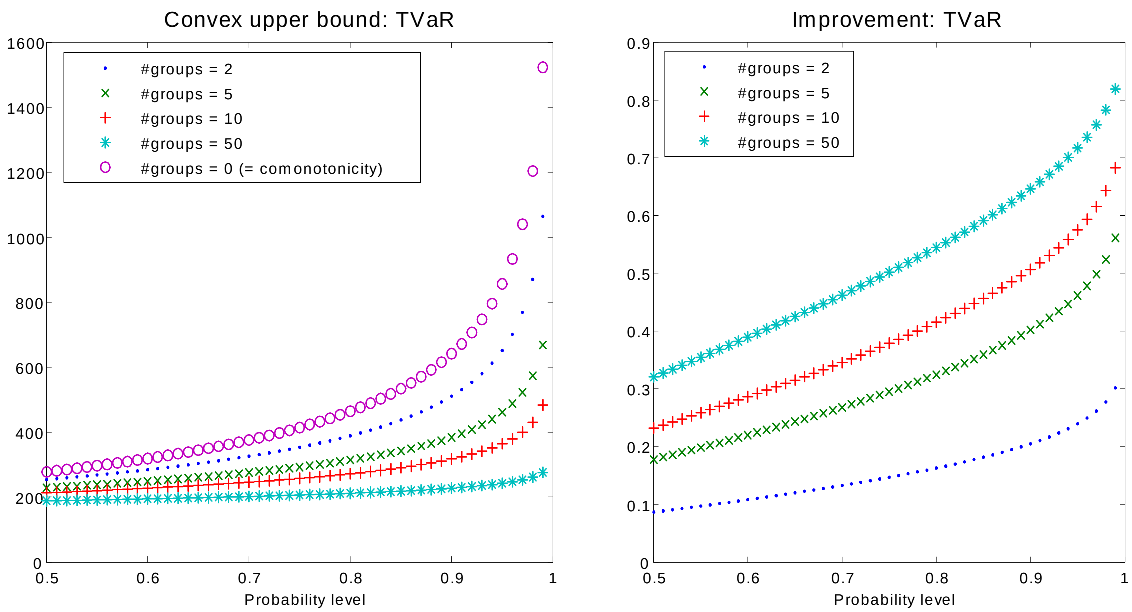

Figure 1: The convex upper bounds for the TVaR in case of different number of independent groups (left) together with the corresponding improvement (right).

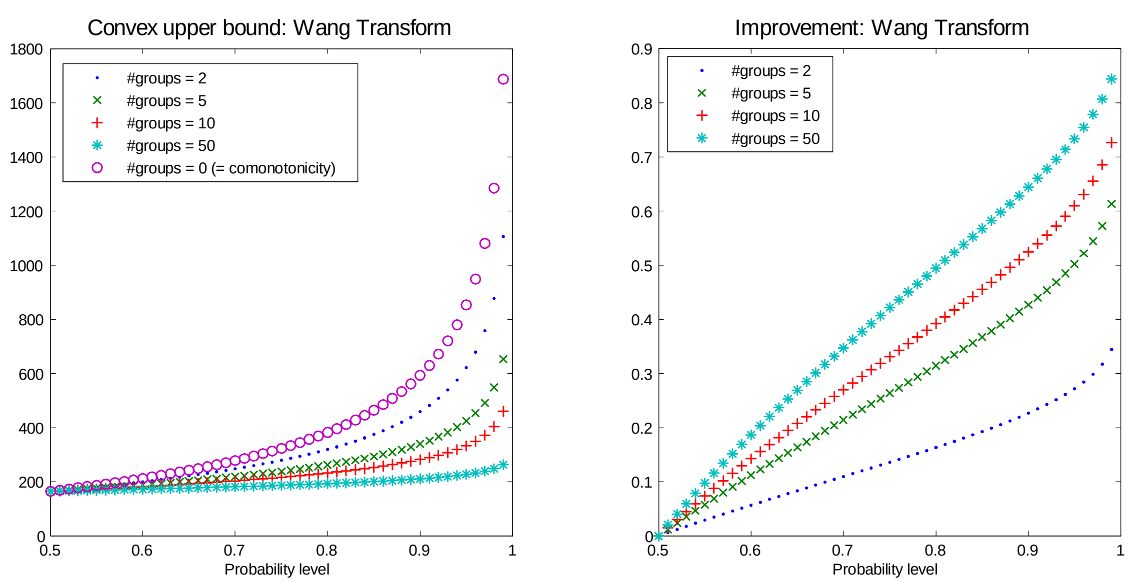

Figure 2: The convex upper bounds for the TVaR in case of different number of independent groups (left) together with the corresponding improvement (right).

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FACULTY OF ECONOMICS AND BUSINESS
Naamsestraat 69 bus 3500
3000 LEUVEN, BELGIË
tel. + 32 16 32 66 12
fax + 32 16 32 67 91
info@econ.kuleuven.be
www.econ.kuleuven.be