Index options in a multivariate Black & Scholes model

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1 Introduction

In this paper, we consider the problem of pricing equity index options (or basket options) in a multivariate Black & Scholes setting. Although this model suffers from some major drawbacks, it pays to consider the pricing of derivatives in the multivariate Black & Scholes model, because it is the most straightforward multivariate extension of the one dimensional Black & Scholes model. Therefore, the multivariate Black & Scholes index option pricing formula can be considered as a benchmark pricing formula, similar to the one-dimensional Black & Scholes formula. A particular application where such a benchmark model plays a crucial role is the study of implied correlation; see e.g. Dhaene, Linders and Schoutens (2013) and Tavin (2013).

In this paper, we derive approximations for the price of an index option using the theory of comonotonicity. Comonotonic random variables are maximal dependent: an increase in one component implies that all components must increase. In the multivariate Black & Scholes model, the index is a weighted sum of dependent lognormal random variables. The distribution of this index is not given in closed form which makes the pricing of index options highly unattractable. We transform the original pricing problem to the pricing of an index option written on a modified index which is a weighted sum of comonotonic lognormal random variables. By choosing this modified index in an appropriate way, one can derive upper and lower bounds for the index option price. It was proven in Chen et al. (2008) that the index option price of a comonotonic index can be decomposed in a linear combination of vanilla option prices for appropriate choosen strike prices. The latter can be determined in closed form using the Black & Scholes formula, which results in an easy and fast algorithm to compute the upper and lower bounds.

In a last step, we combine the upper and lower bound in an approximate value. We prove that the approximate index option curve we obtain, can be interpreted as an index option curve under a synthetic stock market index, where the first two moments of this

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modified index coincide with the first two moments of the real index. Furthermore, we also derive the distribution function of this synthetic index. The idea of valuing an option by replacing the real underlying distribution with a more tractable distribution was already proposed in Jarrow and Rudd (1982).

The pricing of an index option in a Black & Scholes context using the theory of comonotonicity was also considered in Deelstra et al. (2004). Here, the price of an index option is divided in an exact part and a part for which accurate upper and lower bounds can be determined by replacing the original random variable by a convex ordered approximation. In Carmona and Durrleman (2006), the authors derive upper and lower bounds for index options by expressing the index option price as an optimization problem. Both papers provide numerical examples to illustrate the accuracy of the approximations. In this paper, we use the same numerical examples to show that the accuracy of our new approximation is comparable with the existing methods.

There is a vaste literature on approximating the price of a basket or index option. For example, Milevsky and Posner (1998) propose to use the reciprocal gamma distribution to approximate the price of an index option, whereas Hull and White (1993) and Rubinstein (1994) use a binomial tree model. The pricing of basket options using Quasi-Monte Carlo simulation is discussed in Joy et al. (1993). In order to price index options in a more realistic model, Xu and Zheng (2010) derive approximations for the index option price within a jump diffusion model and McWilliams (2011) derives approximations for the index option price in a stochastic delay model.

The paper is organised as follows. In Section 2, we introduce the financial market and the multivariate Black & Scholes model. Furthermore, we recapitulate the notions of convex order and comonotonicity. An analytical formula for the approximated price of an index option is derived in Section 3. Numerical illstrations are given in Section 4. Finally, Section concludes 5 the paper.

2 Options, convex order and comonotonicity

2.1 The financial market

We assume a financial market where n different (dividend or non-dividend paying) stocks, labeled from 1 to n, are traded. Current time is 0, while the time span under consideration is T years. For each stock i, its random price at time t, $0 \leq t \leq T$, is denoted by $X_i(t)$. We denote the stochastic price process of stock i by $\{X_i(t) \mid 0 \leq t \leq T\}$. Hereafter, we will always silently assume that each $X_i(t) \geq 0$ and also that $\mathbb{E}\left[X_i^2(t)\right] < \infty$.

The market index is composed of a linear combination of the n underlying stocks. Denoting the price of the index at time t by $S(t)$, $0 \leq t \leq T$, we have that

$$S(t) = w_1 X_1(t) + w_2 X_2(t) + \ldots + w_n X_n(t), \tag{1}$$

where w_i, $i = 1, 2, \ldots, n$, are positive weights that are fixed up front.

A European call option gives the buyer the right to purchase a stock or an index at a predefined price at a predefined time. For example, a call option written on the index S with maturity T and strike price K has a pay-off at time T of $(S(T) - K)_+$. Its price is denoted by $C[K,T]$. A similar definition exists for a put option, whose price is given by $P[K,T]$.

2.2 The multivariate Black & Scholes model

Assume that the stock prices $X_i(t)$, $i = 1, 2, \ldots, n$, can be described by the following set of SDE's:

$$\frac{\mathrm{d}X_i(t)}{X_i(t)} = \mu_i \mathrm{d}t + \sigma_i \mathrm{d}B_i(t), \text{ for } i = 1, 2, \ldots, n, \tag{2}$$

where $\underline{B}(t) = (B_1(t), B_2(t), \ldots, B_n(t))$ and $\{\underline{B}(t) \mid t \geq 0\}$ is a standard n-dimensional Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$. This probability space is equiped with the filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$ of $\mathcal{F}$ which records the 'the past behavior' of the multivariate Brownian motion. The filtered probability space satisfies the usual technical conditions. The vector $\underline{\mu} = (\mu_1, \mu_2, \ldots, \mu_n)$ contains the drift parameters of each stock. The Variance-Covariance matrix Σ is defined as

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{1,2} & \cdots & \sigma_{1,n} \\ \sigma_{2,1} & \sigma_2^2 & \cdots & \sigma_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n,1} & \sigma_{n,2} & \cdots & \sigma_n^2 \end{pmatrix},$$

where

$$\sigma_{i,j} t = \mathrm{Cov}\left[\sigma_i B_i(t), \sigma_j B_j(t+s)\right]. \tag{3}$$

The correlation $\rho_{i,j}$ between the stocks i and j is given by

$$\rho_{i,j} = \mathrm{Corr}\left[\sigma_i B_i(t), \sigma_j B_j(t+s)\right] \tag{4}$$

and we can write $\sigma_{i,j} = \rho_{i,j}\sigma_i\sigma_j$. We have that $\underline{\mu} \in \mathbb{R}^n$ and $\Sigma \in \mathbb{R}^{n \times n}$. The stock price model described above is called the *multivariate Black & Scholes model*.

We assume that the matrix Σ has full rank. It can be proven that in this case the market is complete and free of arbitrage. Furthermore, there always exists a unique equivalent martingale measure $\mathbb{Q}$. If we replace each μ_i by r in (2) we obtain the set of SDE's describing the stock price dynamics under the risk-neutral probability measure $\mathbb{Q}$. Here, r is the risk-free rate, which is assumed to be known at time $t = 0$ and constant over time. Under this risk-neutral pricing measure, the stock prices at time T are following a lognormal distribution:

$$\ln \frac{X_i(T)}{X_i(0)} \overset{\mathbb{Q}}{=} N\left(\left(r - \frac{1}{2}\sigma_i^2\right)T, \sigma_i^2 T\right), \text{ for } i = 1, 2, \ldots, n, \tag{5}$$

where '$\overset{\mathbb{Q}}{=}$' denotes an 'equality in distribution under the $\mathbb{Q}$-measure'. For a detailed discussion about conditions for completeness and no-arbitrage in the multivariate Black

& Scholes model, we refer to Dhaene, Kukush and Linders (2013), Björk (1998) and the references therein.

The current price of any pay-off at time T can be represented as the discounted expectation of this pay-off. In this price-recipe, discounting is performed using r, whereas expectations are taken with respect to $\mathbb{Q}$. The price of a call option written on stock i, with strike K and maturity T is denoted by $C_i[K,T]$. The price of a put option with the same specifations is denoted by $P_i[K,T]$. Call and put options written on stock i can be expressed as discounted expectations:

$$C_i[K,T] = \mathrm{e}^{-rT}\mathbb{E}\left[(X_i(T)-K)_+\right], \tag{6}$$

$$P_i[K,T] = \mathrm{e}^{-rT}\mathbb{E}\left[(K-X_i(T))_+\right]. \tag{7}$$

If the risk-neutral dynamics of the stock price $X_i(T)$ can be described by the lognormal distribution (5), the option prices $C_i[K,T]$ and $P_i[K,T]$ can be expressed as

$$C_i[K,T] = X_i(0)\,\Phi(d_{i,1}) - K\mathrm{e}^{-rT}\Phi(d_{i,2}), \tag{8}$$

$$P_i[K,T] = K\mathrm{e}^{-rT}\Phi(-d_{i,2}) - X_i(0)\,\Phi(-d_{i,1}), \tag{9}$$

with

$$d_{i,1} = \frac{\left(r+\frac{1}{2}\sigma_i^2\right)T - \ln\frac{K}{X_i(0)}}{\sigma_i\sqrt{T}},$$

$$d_{i,2} = d_{i,1} - \sigma_i\sqrt{T}.$$

Expressions (8) and (9) are the well-known Black & Scholes option pricing formulae; see e.g. Black and Scholes (1973).

In the remainder of this text, expectations (distributions) of functions of the random vector $(X_1(T),\ldots,X_n(T))$ have to be understood as expectations (distributions) under the $\mathbb{Q}$-measure. We will often call them risk-neutral expectations (distributions). Furthermore, the notations $F_{X_i(T)}$ and $F_{S(T)}$ will be used for the time-0 cumulative distribution functions (cdf's) of $X_i(T)$ and $S(T)$ under $\mathbb{Q}$.

In order to avoid unnecessary overloading of the notations, hereafter we will omit the fixed time index T when no confusion is possible. For example, we will write X_i, $C_i[K]$ and $F_{X_i}(x)$ for $X_i(T)$, $C_i[K,T]$ and $F_{X_i(T)}(x)$, respectively.

2.3 Convex order and comonotonicity

In this section we summarize some definitions and results concerning convex order, inverse distributions and comonotonicity needed afterwards.

A r.v. X is said to precede a r.v. Y in *convex order sense*, notation $X \preceq_{cx} Y$, if

$$\begin{cases} \mathbb{E}\left[(X-K)_+\right] \leq \mathbb{E}\left[(Y-K)_+\right] \\ \mathbb{E}\left[(K-X)_+\right] \leq \mathbb{E}\left[(K-Y)_+\right] \end{cases}, \qquad \text{for all } K \in \mathbb{R}. \tag{10}$$

If X and Y are two r.v.'s such that $X \preceq_{cx} Y$, then $\mathbb{E}[X] = \mathbb{E}[Y]$, but Y has heavier (upper and lower) tails than X.

The usual inverse F_X^{-1} of the cdf F_X of a r.v. X is defined by

$$F_{X_i}^{-1}(p) = \inf\{x \in \mathbb{R} \mid F_{X_i}(x) \geq p\}, \qquad p \in [0,1], \tag{11}$$

with $\inf \emptyset = +\infty$, by convention.

The weighted sum S is defined by

$$S = \sum_{i=1}^{n} w_i X_i, \tag{12}$$

where $w_i > 0$. Assume that the marginal stop-loss premiums $\mathbb{E}\left[(X_i - K)_+\right]$ can be determined for any K. Even if we have full information about the marginal distributions, calculating the stop-loss premium $\mathbb{E}\left[(S - K)_+\right]$ is not straightforward as it requires information about the dependence among the marginals. Specifying this dependence structure can be done by choosing an appropriate copula, but the corresponding distribution of S is in most situations unknown or will be too cumbersome to work with.

The random vector $(X_1, \ldots, X_n)$ is said to be *comonotonic* if

$$(X_1, \ldots, X_n) \stackrel{\mathrm{d}}{=} \left(F_{X_1}^{-1}(U), \ldots, F_{X_n}^{-1}(U)\right), \tag{13}$$

where U is a uniform $(0,1)$ r.v. and '$\stackrel{\mathrm{d}}{=}$' denotes 'equality in distribution'. If S is a sum of comonotonic random variables, the stop-loss premium $\mathbb{E}\left[(S - K)_+\right]$ can be decomposed in stop-loss premiums of the marginals with appropriate chosen retentions. We state this result in Theorem 1. For a proof of this theorem, we refer to Kaas et al. (2000). Theorem 1 plays a crucial role in Section 3, where we search for an accurate pricing formula for the index option prices $C[K,T]$ and $P[K,T]$ in a multivariate Black & Scholes model.

Theorem 1 (Decomposition formula) *Consider a comonotonic random vector $(X_1, X_2, \ldots, X_n)$ and denote the weighted sum by S. Assume that F_S is continuous and strictly increasing. For $K \in \left(F_S^{-1}(0), F_S^{-1}(1)\right)$, the stop-loss premium $\mathbb{E}\left[(S - K)_+\right]$ can be decomposed into a linear combination of stop-loss premiums of the marginals involved:*

$$\mathbb{E}\left[(S - K)_+\right] = \sum_{i=1}^{n} w_i \mathbb{E}\left[(X_i - K_i^*)_+\right], \tag{14}$$

where

$$K_i^* = F_{X_i}^{-1}(F_S(K)), \qquad i = 1, \ldots, n, \tag{15}$$

and $F_S(K)$ satisfies the following relation:

$$\sum w_i K_i^* = K. \tag{16}$$

In case the cdf F_S is not continuous and strictly increasing, a similar decomposition formula (14) can be proven for the stop-loss premium $\mathbb{E}\left[(S-K)_+\right]$ of a comonotonic sum S, but now the expression for the strike price K_i^* will be slightly different. A sufficient condition for F_S to be strictly increasing and continuous is that the marginal cdfs F_{X_i} are strictly increasing and continuous. This condition is especially met when dealing with lognormal r.v.'s. Furthermore, for appropriate choosen K_i^*, the decompisition formula (14) remains to hold when $K \notin \left(F_S^{-1}(0), F_S^{-1}(1)\right)$; see .e.g Dhaene et al. (2002a) and Chen et al. (2013).

For an extensive overview of the theory of comonotonicity, including proofs of the results mentioned in this subsection, we refer to Dhaene et al. (2002a). Financial and actuarial applications of the concept of comonotonicity are described in Dhaene et al. (2002b). An updated overview of applications of comonotonicity can be found in Deelstra et al. (2011).

3 Convex approximations for index options

In this section we derive the approximate index option prices $\overline{C}[K]$ and $\overline{P}[K]$ for $C[K]$ and $P[K]$, respectively. Furthermore, we show that the curves $\overline{C}$ and $\overline{P}$ can be considered as index option curves written on a synthetic market index $\overline{S}$, which serves as an approximate index for the real index S; see Theorem 11. The approximate index option price $\overline{C}[K]$ is a linear combination of the upper bound $C^c[K]$ and the lower bound $C^l[K]$, where the interpolation weight is chosen such that $\mathrm{Var}[S] = \mathrm{Var}\left[\overline{S}\right]$. Similarly, $\overline{P}[K]$ is a linear combination of the upper bound $P^c[K]$ and the lower bound $P^l[K]$, where we use the same interpolation weights. Note that using only the upper or lower bound is not desirable as this will lead to a consistent over or under estimation of the real index option price.

3.1 Upper bound

In this subsection, we replace the real sum S by the random sum S^c, which is defined as

$$S^c = w_1 F_{X_1}^{-1}(U) + \ldots + w_n F_{X_n}^{-1}(U). \tag{17}$$

The index S^c is called the *comonotonic stock market index* and it is, like the index S, a weighted average of the marginals $X_1, X_2, \ldots X_n$, but the dependence structure is assumed to be comonotonic. In Kaas et al. (2000) it is proven that the comonotonic sum S^c is a convex upper bound for the sum S :

$$S \preceq_{cx} S^c. \tag{18}$$

Consider the pay-offs $(S^c-K)_+$ and $(K-S^c)_+$ at time T. These pay-offs can be interpreted as pay-offs of an index call and put option written on a stock market index that can be described by S^c. Note, however, that these options are not traded actively and its prices cannot be observed in the market, because the stock market index S is in general not equal to the comonotonic stock market index S^c. If we denote the theoretical prices of these synthetic index options by $C^c[K]$ and $P^c[K]$, we can determine them as:

$$C^c[K] = \mathrm{e}^{-rT}\mathbb{E}\left[(S^c - K)_+\right], \qquad (19)$$
$$P^c[K] = \mathrm{e}^{-rT}\mathbb{E}\left[(K - S^c)_+\right].$$

For the comonotonic index option prices, we can prove the put-call parity

$$C^c[K] = P^c[K] - \mathrm{e}^{-rT}K + \mathrm{e}^{-rT}\mathbb{E}[S]. \qquad (20)$$

From expression (17) we find that the components of S^c are all non-decreasing functions of the same r.v. U. Therefore, we can interprete the index S^c as a 'worst case' scenario. All stocks composing the index will go simultaneously up or simultaneously down. As a result, the price of an index option written on S^c is an upper bound for the real index option price.

Theorem 2 *The index call and put option prices $C[K]$ and $P[K]$ are constrained from above by $C^c[K]$ and $P^c[K]$, respectively:*

$$C[K] \leq C^c[K], \text{ for all } K \geq 0,$$
$$P[K] \leq P^c[K], \text{ for all } K \geq 0.$$

Proof. This is a direct consequence of the convex order relation (18) and the characterization of convex order; see (10).

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Theorem 2 holds regardless the assumption about the marginal distributions F_{X_i}. In Chen et al. (2008) and Hobson et al. (2005), model-free upper bounds for index options are derived using Theorem 2 together with Theorem 1. Furthermore, it is shown that there exists an optimal static super-replicating strategy for an index option, which consists in buying a linear combination of vanilla options. Additional details and computational issues are given in Chen et al. (2013) and Linders et al. (2012).

In this section, we specified the marginal distributions to be lognormally distributed. In this special case, we can determine S^c explicitly in terms of the marginal volatilities σ_i, the risk-free rate r and the cdf Φ of a standard normal distribution.

Theorem 3 (A closed form expression for S^c.) *Consider a market where the assets follow the multivariate Black & Scholes model (2). The comonotonic market index S^c is given by the following expression:*

$$S^c \stackrel{\mathbb{Q}}{=} \sum_{i=1}^{n} w_i X_i(0) \exp\left\{\left(r - \frac{\sigma_i^2}{2}\right)T + \sigma_i\sqrt{T}\Phi^{-1}(U)\right\}, \qquad (21)$$

where Φ denotes the cdf of a standard normal random variable and U denotes a r.v. which is uniformly distributed on the unit interval. Its variance is given by

$$\mathrm{Var}[S^c] = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j X_i(0) X_j(0) \mathrm{e}^{2rT}\left(\mathrm{e}^{\sigma_i\sigma_j T} - 1\right). \qquad (22)$$

Proof. The marginal risk-neutral distributions are given by (5). If we combine this expression with Theorem 1 in Dhaene et al. (2002a), the inverse cdf $F_{X_i}^{-1}$ is given by

$$F_{X_i}^{-1}(p) = X_i(0)\exp\left\{\left(r - \frac{\sigma_i^2}{2}\right)T + \sigma_i\sqrt{T}\Phi^{-1}(p)\right\}. \tag{23}$$

Taking into account the definition of S^c in (17) proves (21).

We write the variance $\mathrm{Var}[S^c]$ as

$$\mathrm{Var}[S^c] = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \mathrm{Cov}\left[F_{X_i}^{-1}(U), F_{X_j}^{-1}(U)\right].$$

We have that

$$\begin{aligned}\mathrm{Cov}\left[F_{X_i}^{-1}(U), F_{X_j}^{-1}(U)\right] &= X_i(0)\,X_j(0)\,\mathrm{e}^{2rT-\frac{1}{2}\left(\sigma_i^2+\sigma_j^2\right)T} \\ &\quad\times \mathrm{Cov}\left[\mathrm{e}^{\sigma_i\sqrt{T}\Phi^{-1}(U)},\ \mathrm{e}^{\sigma_j\sqrt{T}\Phi^{-1}(U)}\right].\end{aligned}$$

Note that if $\sigma \in \mathbb{R}$, then $\mathbb{E}\left[\mathrm{e}^{\sigma\Phi^{-1}(U)}\right] = \mathrm{e}^{\frac{\sigma^2}{2}}$ and

$$\mathrm{Cov}\left[\mathrm{e}^{\sigma_i\sqrt{T}\Phi^{-1}(U)},\ \mathrm{e}^{\sigma_j\sqrt{T}\Phi^{-1}(U)}\right] = \mathrm{e}^{\frac{1}{2}\left(\sigma_i^2+\sigma_j^2\right)T}\left(\mathrm{e}^{\sigma_i\sigma_j T} - 1\right),$$

for each pair $i, j = 1, 2, \ldots n$, which proves (22).

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In the following theorem, we prove that the upper bound $C^c[K]$ for the index call option and the upper bound $P^c[K]$ for the index put option can be expressed in terms of vanilla call and put option prices on the components of S.

Theorem 4 *For $K \geq 0$, the prices $C^c[K]$ and $P^c[K]$ of the index options with pay-off at time T given by $(S^c - K)_+$ and $(K - S^c)_+$, respectively, can be expressed as follows:*

$$C^c[K] = \sum_{i=1}^{n} w_i C_i[K_i^*], \tag{24}$$

$$P^c[K] = \sum_{i=1}^{n} w_i P_i[K_i^*], \tag{25}$$

where

$$K_i^* = X_i(0)\exp\left\{\left(r - \frac{\sigma_i^2}{2}\right)T + \sigma_i\sqrt{T}\Phi^{-1}\left(F_{S^c}(K)\right)\right\}, \tag{26}$$

and $F_{S^c}(K)$ is determined using the relation

$$\sum_{i=1}^{n} w_i K_i^* = K. \tag{27}$$

Proof. The sum S^c is a sum of comonotonic r.v.'s. Furthermore, the marginal cdf's F_{X_i} are strictly increasing and continuous for all $i = 1, 2, \ldots, n$. It can be proven that F_{S^c} is also continuous and strictly increasing; see e.g. Dhaene et al. (2002a). From (21), it follows that $\left(F_{S^c}^{-1}(0), F_{S^c}^{-1}(1)\right) = (0, +\infty)$. Combining (19) and (6) with Theorem 1 results in (24). The put-call parity (20) proves expression (25). The choice (26) for K_i^* follows from (23) and (15).

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The right hand side of (24) is a linear combination of vanilla call options and the right hand side of (25) is a linear combination of vanilla put options. Using the Black & Scholes option pricing formula we can find an analytical expression for the prices $C^c[K]$ and $P^c[K]$.

Theorem 5 *For $K \geq 0$, the prices $C^c[K]$ and $P^c[K]$ of the index options with pay-off at time T given by $(S^c - K)_+$ and $(K - S^c)_+$, respectively, can be expressed as follows:*

$$C^c[K] = \sum_{i=1}^{n} w_i \left(X_i(0)\,\Phi(d_{i,1}) - K_i^* \mathrm{e}^{-rT} \Phi(d_{i,2})\right), \tag{28}$$

$$P^c[K] = \sum_{i=1}^{n} w_i \left(K_i^* \mathrm{e}^{-rT} \Phi(-d_{i,2}) - X_i(0)\,\Phi(-d_{i,1})\right) \tag{29}$$

where K_i^* is defined in (26) of Theorem 4 and

$$d_{i,1} = \frac{\ln \frac{X_i(0)}{K_i^*} + \left(r + \frac{\sigma_i^2}{2}\right) T}{\sigma_i \sqrt{T}},$$
$$d_{i,2} = d_{i,1} - \sigma_i \sqrt{T}.$$

Proof. Using expressions (8) and (9) in Theorem 4 proves the result.

■

3.2 Lower bound

In this subsection we replace the market index S by the conditional sum S^l, which is defined as follows:

$$S^l = w_1 \mathbb{E}[X_1 \mid \Lambda] + \ldots + w_n \mathbb{E}[X_n \mid \Lambda],$$

where

$$\Lambda = \sum_{i=1}^{n} \lambda_i \ln \frac{X_i}{X_i(0)}, \tag{30}$$

for $\lambda_i > 0$. In Kaas et al. (2000) it is proven that the sum S^l is a convex lower bound for the sum S :

$$S^l \preceq_{cx} S. \tag{31}$$

Consider the pay-offs $(S_l - K)_+$ and $(K - S_l)_+$ which have to be paid at time T. These pay-offs can be interpreted as the pay-offs of an index call and put option written on a stock market index which can be described by S^l. The stock market index S will differ from S^l, which makes it impossible to invest in the index S^l. We denote by $C^l[K]$ and $P^l[K]$ the prices of the synthethic index call and put option written on S^l. The theoretical price of these options are given by

$$\begin{aligned} C^l[K] &= \mathrm{e}^{-rT}\mathbb{E}\left[\left(S^l - K\right)_+\right], \\ P^l[K] &= \mathrm{e}^{-rT}\mathbb{E}\left[\left(K - S^l\right)_+\right]. \end{aligned} \tag{32}$$

We can prove the put-call parity for these option prices:

$$C^l[K] = P^l[K] - \mathrm{e}^{-rT}K + \mathrm{e}^{-rT}\mathbb{E}[S]. \tag{33}$$

Theorem 6 *The index call and put option prices $C[K]$ and $P[K]$ are constrained from below by $C^l[K]$ and $P^l[K]$, respectively:*

$$\begin{aligned} C^l[K] &\leq C[K], \text{ for all } K \geq 0, \\ P^l[K] &\leq P[K], \text{ for all } K \geq 0. \end{aligned}$$

Proof. This is a direct consequence of the convex order relation (31).

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If the marginals are lognormal distributed, an analytical expression for S^l in terms of the r.v. Λ, the risk-free rate r, the maturity T and the marginal volatilities σ_i can be derived.

Theorem 7 *Consider a market where the assets follow the multivariate Black & Scholes model (2). The convex lower bound S^l can be expressed as follows:*

$$S^l \stackrel{\mathbb{Q}}{=} \sum_{i=1}^{n} w_i X_i(0) \exp\left\{\left(r - \frac{\sigma_i^2}{2}r_i^2\right)T + r_i\sigma_i\sqrt{T}\Phi^{-1}(U)\right\}. \tag{34}$$

In this formula, $r_i = \operatorname{Corr}\left[\ln\frac{X_i}{X_i(0)}, \Lambda\right]$ *and*

$$\Lambda \stackrel{\mathbb{Q}}{=} \sum_{i=1}^{n} \lambda_i \left(\left(r - \frac{\sigma_i^2}{2}\right)T + \sigma_i\sqrt{T}B_i(1)\right). \tag{35}$$

The variance is given by

$$\operatorname{Var}\left[S^l\right] = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i X_i(0) w_j X_j(0) \mathrm{e}^{2rT}\left(\mathrm{e}^{r_i r_j \sigma_i \sigma_j T} - 1\right). \tag{36}$$

Proof. Note that $B_i(T) \stackrel{\mathrm{d}}{=} \sqrt{T} N(0,1)$, for $i = 1, 2, \ldots, n$. From (5), we find that the risk-neutral dynamics of the logreturns are given by

$$\ln \frac{X_i}{X_i(0)} \stackrel{\mathbb{Q}}{=} \left(r - \frac{1}{2}\sigma_i^2\right) T + \sigma_i B_i(T).$$

Combining this expression with (30), we find that (35) holds. Furthermore, Λ has a normal distribution with mean $\mu_\Lambda T$ and variance $\sigma_\Lambda^2 T$. Remember that for a bivariate normal distribution (X, Y) with $\rho = \text{Corr}[X, Y]$, we have that $X \mid Y$ has again a normal distribution with mean:

$$\mathbb{E}[X \mid Y] = \mathbb{E}[X] + \rho \sqrt{\frac{\text{Var}[X]}{\text{Var}[Y]}} (Y - \mathbb{E}[Y]), \tag{37}$$

and variance $\text{Var}[X](1 - \rho^2)$. Using expression (37), we find that $\ln \frac{X_i}{X_i(0)} \mid \Lambda$ has a normal distribution with mean

$$\mathbb{E}\left[\ln \frac{X_i}{X_i(0)} \mid \Lambda\right] = \left(r - \frac{\sigma_i^2}{2}\right) T + r_i \sigma_i \sqrt{T} \left(\frac{\Lambda - \mathbb{E}[\Lambda]}{\sqrt{\text{Var}[\Lambda]}}\right),$$

and variance

$$\text{Var}\left[\ln \frac{X_i}{X_i(0)} \mid \Lambda\right] = \sigma_i^2 T \left(1 - r_i^2\right).$$

Finally, the equality $\left(\frac{\Lambda - \mathbb{E}[\Lambda]}{\sqrt{\text{Var}[\Lambda]}}\right) \stackrel{\mathrm{d}}{=} \Phi^{-1}(U)$ proves (34).

The proof of (36) follows the same lines as the proof of (22). ■

Remark 8 (Calculation of r_i) *The variance of Λ and X_j are both involved in the calculation of r_i. The variance of Λ is denoted by $\sigma_\Lambda^2 T$. Using (4) and (35), we find that*

$$\sigma_\Lambda^2 = \sum_{i=1}^{n} \lambda_i^2 \sigma_i^2 + 2 \sum_{\substack{i=1 \\ j<i}}^{n} \lambda_i \lambda_j \rho_{i,j} \sigma_i \sigma_j.$$

Plugging the variance σ_Λ^2 in the formula for the correlation r_i results in

$$r_i = \text{Corr}\left[\ln \frac{X_i}{X_i(0)}, \Lambda\right] \tag{38}$$

$$= \frac{\text{Cov}\left[\ln \frac{X_i}{X_i(0)}, \Lambda\right]}{\sqrt{\text{Var}\left[\ln \frac{X_i}{X_i(0)}\right] \text{Var}[\Lambda]}} \tag{39}$$

$$= \frac{T \sum_{j=1}^{n} \lambda_j \text{Cov}[\sigma_i B_i(1), \sigma_j B_j(1)]}{T \sigma_i \sigma_\Lambda}$$

$$= \frac{\sum_{j=1}^{n} \lambda_j \rho_{i,j} \sigma_j}{\sigma_\Lambda}. \tag{40}$$

Given that $\rho_{i,j} \geq 0$, all correlations r_i are positive.

∇

The sum S^l is a comonotonic sum if all r_i are non-negative. In Deelstra et al. (2004) it is proven that there always exists λ_i, $i = 1, 2, \ldots, n$ in (30) such that S^l is a comonotonic sum. However, for sake of simplicity, we make the following assumption:

$$\underline{\textbf{Assumption}}: \ \rho_{i,j} > 0 \text{ for all } i, j = 1, 2, \ldots, n. \tag{41}$$

Under this assumption, the market index S^l can be expressed as a sum of n comonotonic lognormal random variables V_i:

$$S^l \overset{\mathbb{Q}}{=} \sum_{i=1}^{n} w_i V_i,$$

where $V_i \overset{\mathbb{Q}}{=} X_i(0) \exp\left\{\left(r - \frac{\sigma_i^2}{2} r_i^2\right) T + r_i \sigma_i \sqrt{T} \Phi^{-1}(U)\right\}$ can be considered as an adjusted stock price process for stock i. Each r.v. V_i is an increasing function of the same r.v. U. Under the adjusted price process $\{V_i(t) \mid t \geq 0\}$, the price of stock i at time T is again lognormal distributed

$$\ln \frac{V_i}{V_i(0)} \overset{\mathbb{Q}}{=} N\left(\left(r - \frac{\sigma_i^2}{2} r_i^2\right) T, r_i^2 \sigma_i^2 T\right). \tag{42}$$

The following theorem states that the prices $C^l[K]$ and $P^l[K]$ can be expressed as a weighted sum of n vanilla option prices, written on the adjusted stock prices V_i.

Theorem 9 *Consider a market where the assets follow the multivariate Black & Scholes model (2), where $\rho_{i,j} \geq 0$, for all i, j. The prices $C^l[K]$ and $P^l[K]$ for $K \geq 0$ of the index options with pay-off at time T given by $\left(S^l - K\right)_+$ and $\left(K - S^l\right)_+$, respectively, can be expressed as follows:*

$$C^l[K] = \sum_{i=1}^{n} w_i \mathrm{e}^{-rT} \mathbb{E}\left[(V_i - K_i^*)_+\right], \tag{43}$$

$$P^l[K] = \sum_{i=1}^{n} w_i \mathrm{e}^{-rT} \mathbb{E}\left[(K_i^* - V_i)_+\right], \tag{44}$$

where

$$K_i^* = X_i(0) \exp\left\{\left(r - \frac{\sigma_i^2}{2} r_i^2\right) T + r_i \sigma_i \sqrt{T} \Phi^{-1}\left(F_{S^l}(K)\right)\right\},$$

and $F_{S^l}(K)$ is determined using

$$\sum_{i=1}^{n} w_i K_i^* = K.$$

Proof. The assumption (41) assures that S^l is a sum of the comonotonic r.v.'s $V_1, V_2, \ldots, V_n$. Each V_i has a lognormal distribution and from Theorem 1 in Dhaene et al. (2002a), we find that

$$F_{V_i}^{-1}(p) = X_i(0) \exp\left\{\left(r - \frac{r_i \sigma_i^2}{2}\right) T + r_i \sigma_i \sqrt{T} \Phi^{-1}(p)\right\}. \tag{45}$$

The sum S^l is a sum of comonotonic r.v.'s. Furthermore, the marginals F_{V_i} are strictly increasing and continuous for all $i = 1, 2, \ldots, n$. So the cdf F_{S^l} is also continuous and strictly increasing. From (34), if follows that $\left(F_{S^l}^{-1}(0), F_{S^l}^{-1}(1)\right) = (0, +\infty)$. Finally, combining (32) and (6) with Theorem 1 proves (43). Applying the put-call parity (33) proves (44).

■

Theorem 10 *Consider a market where the assets follow the multivariate Black & Scholes model (2), where $\rho_{i,j} \geq 0$, for all i, j. The prices $C^l[K]$ and $P^l[K]$ for $K \geq 0$ of the index options with pay-off at time T given by $\left(S^l - K\right)_+$ and $\left(K - S^l\right)_+$, respectively, can be expressed as follows:*

$$C^l[K] = \sum_{i=1}^{n} w_i \left(X_i(0)\,\Phi(d_{i,1}) - K_i^* \mathrm{e}^{-rT} \Phi(d_{i,2})\right), \tag{46}$$

$$P^l[K] = \sum_{i=1}^{n} w_i \left(K_i^* \mathrm{e}^{-rT} \Phi(-d_{i,2}) - X_i(0)\,\Phi(-d_{i,1})\right), \tag{47}$$

where the K_i^* is defined as in Theorem 9 and

$$d_{i,1} = \frac{\ln \frac{X_i(0)}{K_i^*} + \left(r + \frac{\sigma_i^2}{2} r_i^2\right) T}{r_i \sigma_i \sqrt{T}},$$
$$d_{i,2} = d_{i,1} - r_i \sigma_i \sqrt{T}.$$

Proof. The marginals V_i, $i = 1, 2, \ldots, n$ have a lognormal distribution; see (42). Combining this observation with (8) and (9) results in a closed form expression for $\mathrm{e}^{-rT}\mathbb{E}\left[(V_i - K_i^*)_+\right]$ and $\mathrm{e}^{-rT}\mathbb{E}\left[(K_i^* - V_i)_+\right]$ for $i = 1, 2, \ldots, n$. Using these expressions in (43) and (44) proves the result.

■

3.2.1 On the choice of λ_i

In Cheung et al. (2013) it is proven that for a sufficiently nice convex function[1] u, we have that $\mathbb{E}\left[u\left(S^l\right)\right] \leq \mathbb{E}\left[u(S)\right]$. Moreover, if u is strictly convex $\mathbb{E}\left[u\left(S^l\right)\right] = \mathbb{E}\left[u(S)\right]$ is equivalent with $S^l \stackrel{\mathrm{d}}{=} S$. Therefore, it is reasonable to take λ_i such that $\mathbb{E}\left[u\left(S^l\right)\right]$ is as close as possible to $\mathbb{E}\left[u(S)\right]$, for a particular strictly convex function u.

Here, we choose u to be equal to $u(x) = (x - \mathbb{E}[S])^2$. Then $\mathbb{E}[u(S)] = \mathrm{Var}[S]$. Following the ideas of Vanduffel et al. (2005), we approximate the variance of S^l as follows:

$$\mathbb{E}\left[u\left(S^l\right)\right] = \mathrm{Var}\left[S^l\right] \approx \sum_{i=1}^{n}\sum_{j=1}^{n} w_i X_i(0)\, w_j X_j(0)\, \mathrm{e}^{2rT} r_i r_j \sigma_i \sigma_j T. \tag{48}$$

[1] A function u is sufficiently nice if has an absolutely continuous derivative u'.

Remember that $r_i = \mathrm{Corr}\left[\ln \frac{X_i}{X_i(0)}, \Lambda\right]$ and $\mathrm{Var}[\Lambda] = \sigma_\Lambda^2 T$. Then we can write

$$r_i \sigma_i \sqrt{T} = \frac{\mathrm{Cov}\left[\ln \frac{X_i}{X_i(0)}, \Lambda\right]}{\sigma_\Lambda \sqrt{T}}.$$

Using this relation, the right hand side of (48) becomes

$$\sum_{i=1}^{n} \sum_{j=1}^{n} w_i X_i\left(0\right) w_j X_j\left(0\right) \mathrm{e}^{2rT} \frac{\mathrm{Cov}\left[\ln \frac{X_i}{X_i(0)}, \Lambda\right] \mathrm{Cov}\left[\ln \frac{X_j}{X_j(0)}, \Lambda\right]}{\sigma_\Lambda^2 T}.$$

which can be written as

$$\frac{\left(\mathrm{Cov}\left[\sum_{i=1}^{n} w_i X_i\left(0\right) \mathrm{e}^{rT} \ln \frac{X_i}{X_i(0)}, \Lambda\right]\right)^2}{\sigma_\Lambda^2 T}.$$

Finally, we can approximate the variance as follows:

$$\begin{aligned}\mathbb{E}\left[u\left(S^l\right)\right] = \mathrm{Var}\left[S^l\right] \approx & \left(\mathrm{Corr}\left[\sum_{i=1}^{n} w_i X_i\left(0\right) \mathrm{e}^{rT} \ln \frac{X_i}{X_i\left(0\right)}, \Lambda\right]\right)^2 \\ & \times \mathrm{Var}\left[\sum_{i=1}^{n} w_i X_i\left(0\right) \mathrm{e}^{rT} \ln \frac{X_i}{X_i\left(0\right)}\right].\end{aligned} \quad (49)$$

If $\mathbb{E}\left[u\left(S^l\right)\right]$ reaches its maximal value, $\mathbb{E}\left[u\left(S^l\right)\right]$ is as close as possible to $\mathbb{E}\left[u\left(S\right)\right]$. The right hand side of (49) is maximal if we take Λ such that

$$\mathrm{Corr}\left[\sum_{i=1}^{n} w_i X_i\left(0\right) \mathrm{e}^{rT} \ln \frac{X_i}{X_i\left(0\right)}, \Lambda\right] = 1.$$

So we find that a globally optimal choice for Λ is

$$\Lambda = \sum_{i=1}^{n} w_i X_i\left(0\right) \mathrm{e}^{rT} \ln \frac{X_i}{X_i\left(0\right)},$$

hence

$$\lambda_i = w_i X_i\left(0\right) \mathrm{e}^{rT}, \quad \text{for } i = 1, 2, \ldots, n. \quad (50)$$

3.3 Moments based approximation

The upper and lower bounds derived in Theorems 5 and 10 can be combined in one approximate value for the prices $C\left[K\right]$ and $P\left[K\right]$, which we will denote by $\overline{C}\left[K\right]$ and

$\overline{P}[K]$, respectively. This approximation will be a linear combination of the convex upper and lower bound, using a factor $z \in [0,1]$:

$$\overline{C}[K] = zC^{l}[K] + (1-z)\,C^{c}[K]\,, \text{ for all } K \geq 0, \tag{51}$$

$$\overline{P}[K] = zP^{l}[K] + (1-z)\,P^{c}[K]\,, \text{ for all } K \geq 0. \tag{52}$$

The non-increasing convex curve $\overline{C}$ and the non-decreasing convex curve $\overline{P}$ can be interpreted as option curves of a synthetic stock market index $\overline{S}$. The index $\overline{S}$ is not traded in the market, but the theoretical price of an index option on $\overline{S}$ is given by

$$\overline{C}[K] = \mathrm{e}^{-rT}\mathbb{E}\left[\left(\overline{S}-K\right)_{+}\right], \text{ for all } K \geq 0, \tag{53}$$

$$\overline{P}[K] = \mathrm{e}^{-rT}\mathbb{E}\left[\left(K-\overline{S}\right)_{+}\right], \text{ for all } K \geq 0. \tag{54}$$

The cdf $F_{\overline{S}}$ can be expressed in terms of the cdf's F_{S^l} and F_{S^c}.

Theorem 11 *Consider a market where the assets follow the multivariate Black & Scholes model (2), where $\rho_{i,j} \geq 0$, for all i,j. Assume that the prices for call and put options written on the index $\overline{S}$ are given by (51) and (52), respectively. Then we have that the cdf $F_{\overline{S}}$ of $\overline{S}$ is given by*

$$F_{\overline{S}}(x) = zF_{S^l}(x) + (1-z)\,F_{S^c}(x)\,, \text{ for all } x \in \mathbb{R}.$$

Proof. The curve $\overline{C}$ is a call option curve written on $\overline{S}$, so (53) must hold. Then the cdf $F_{\overline{S}}$ is fully determined by the option curve $\overline{C}$ via the relation

$$F_{\overline{S}}(x) = 1 + \mathrm{e}^{rT}\overline{C}'[x+]\,,$$

where $\overline{C}'[x+]$ denotes the right derivative of $\overline{C}$ in x; see e.g. Breeden and Litzenberger (1978). Applying this relation in (51) proves (53). Relation (54) follows from the put-call parity.

■

Different values for z will lead to different option curves $\overline{C}$ and $\overline{P}$ and as a result also to different distributions for $\overline{S}$. The value for z in (51) is determined such that $\mathbb{E}\left[u\left(\overline{S}\right)\right] = \mathbb{E}\left[u(S)\right]$. The latter equality cannot be used to conclude that $\overline{S} \stackrel{\mathrm{d}}{=} S$ because the r.v.'s $\overline{S}$ and S are not convex ordered.

Theorem 12 *Consider a market where the assets follow the multivariate Black & Scholes model (2), where $\rho_{i,j} \geq 0$, for all i,j. If in (51) and (52), z is given by*

$$z = \frac{\mathbb{E}\left[u(S^c)\right] - \mathbb{E}\left[u(S)\right]}{\mathbb{E}\left[u(S^c)\right] - \mathbb{E}\left[u(S^l)\right]}, \tag{55}$$

then $\mathbb{E}\left[u\left(\overline{S}\right)\right] = \mathbb{E}\left[u(S)\right]$.

Proof. In Cheung et al. (2013), it is shown that $\mathbb{E}\left[u\left(\overline{S}\right)\right]-\mathbb{E}\left[u\left(S\right)\right]$ can be expressed as

$$\mathbb{E}\left[u\left(\overline{S}\right)\right]-\mathbb{E}\left[u\left(S\right)\right]=\int_0^{\mathbb{E}[S]}u''\left(K\right)\left(\overline{P}\left[K\right]-P\left[K\right]\right)\mathrm{d}K+\int_{\mathbb{E}[S]}^{+\infty}u''\left(K\right)\left(\overline{C}\left[K\right]-C\left[K\right]\right)\mathrm{d}K.$$

Inserting (51) in this expression results in

$$\begin{aligned}\mathbb{E}\left[u\left(\overline{S}\right)\right]-\mathbb{E}\left[u\left(S\right)\right]&=\int_0^{\mathbb{E}[S]}u''\left(K\right)\left(zP^l\left[K\right]+\left(1-z\right)P^c\left[K\right]-P\left[K\right]\right)\mathrm{d}K\\&\quad+\int_{\mathbb{E}[S]}^{+\infty}u''\left(K\right)\left(zC^l\left[K\right]+\left(1-z\right)C^c\left[K\right]-C\left[K\right]\right)\mathrm{d}K\\&=\mathbb{E}\left[u\left(S^c\right)\right]-\mathbb{E}\left[u\left(S\right)\right]-z\left(\mathbb{E}\left[u\left(S^c\right)\right]-\mathbb{E}\left[u\left(S^l\right)\right]\right),\end{aligned}$$

from which we find that $\mathbb{E}\left[u\left(\overline{S}\right)\right]-\mathbb{E}\left[u\left(S\right)\right]=0$ if z is given by (55). ■

Throughout this paper, we will use the choice $u\left(x\right)=\left(x-\mathbb{E}\left[S\right]\right)^2$. Theorem 12 states that if we take

$$z=\frac{\mathrm{Var}\left[S^c\right]-\mathrm{Var}\left[S\right]}{\mathrm{Var}\left[S^c\right]-\mathrm{Var}\left[S^l\right]},\tag{56}$$

then the index option surface is approximated such that $\mathrm{Var}\left[\overline{S}\right]=\mathrm{Var}[S]$. In the sequel of the paper we will use $\overline{C}\left[K\right]$ and $\overline{P}\left[K\right]$ as approximations for the prices of index call and put options:

$$\begin{aligned}\overline{C}\left[K\right]&\approx C\left[K\right],\ \text{for all } K\geq 0,\\\overline{P}\left[K\right]&\approx P\left[K\right],\ \text{for all } K\geq 0.\end{aligned}$$

Convex approximations for sums of dependent lognormal r.v.'s proved to be successful in earlier literature; see e.g. Vanduffel et al. (2005), Dhaene et al. (2005) and Van Weert (2011). The idea of combining an upper and lower bound in an approximate option value was proposed in Vyncke et al. (2004) for the pricing of an Asian option.

4 Numerical illustration

The efficiency of the comonotonic approximations $\overline{C}\left[K\right]$ and $\overline{P}\left[K\right]$ for the option prices $C\left[K\right]$ and $P\left[K\right]$ is discussed in this section with the help of numerical illustrations. We first consider the bivariate case, so $n=2$. The correlation between the stocks is denoted by ρ and $S\left(T\right)=w_1X_1\left(T\right)+w_2X_2\left(T\right)$. The interest rate r is set to 5%. An example with equal weights and another example with unequal weights will be investigated. In each situation, we compare the approximate option prices with the corresponding Monte Carlo estimates, where 10^6 simulated values are used. We determine option prices for the maturities $T=1$ and $T=3$. Note that strike prices are expressed in terms of forward moneyness. A basket strike price K has forward moneyness equal to $\frac{K}{\mathbb{E}[S]}$. We assume that the current prices of the non-dividend paying stocks are given by $X_1\left(0\right)=X_2\left(0\right)=100$ and the weights are equal, $w_1=w_2=0.5$. The results are listed in Table 1.

Table 2 gives the numerical values for the situation where the weights and the initial stock prices are different. Here, we have that $X_1(0) = 130$ and $X_2(0) = 70$. Furthermore, $w_1 = 0.3$ and $w_2 = 0.7$. Both tables show that the approximate values are very close to the simulated values. The two situations considered in Table 1 and 2 are also handled in Deelstra et al. (2004). In this paper, the authors discuss various approximations for the price of an arithmetic basket option, which are also based on comonotonic approximations.

We also consider the pricing of an index option, where the index S is composed of $n = 50$ stocks. For simplicity, we take $r = 0\%$, $T = 1$ and $X_i(0) = 100$, $w_i = 1/50$, for $i = 1, 2, \ldots, 50$. This particular situation was also considered in Carmona and Durrleman (2006). The performance of the approximations $\overline{C}[K]$ is compared with the Monte-Carlo simulation $C^{mc}[K]$ and listed in Table 3. Tables 1-3 show that the approximation $\overline{C}[K]$ is close to the simulated option price $C^{mc}[K]$. Indeed, the error ε_K is defined as $\varepsilon[K] = 1 - \frac{C[K]}{C^{mc}[K]}$ and is always less than 1%.

Table 1: Approximations and simulations: $r = 5\%, w_1 = w_2 = 0.5, X_1(0) = X_2(0) = 100$

K	T	ρ	σ	$C^{mc}[K]$	$\bar{C}[K]$	$\epsilon[K]$
10%OTM						
115.64	1	0.3	0.2	2.9	2.8987	0.04%
			0.4	9.12	9.1119	0.09%
		0.7	0.2	3.72	3.7189	0.03%
			0.4	10.88	10.8794	0.01%
127.80	3	0.3	0.2	7.39	7.3835	0.09%
			0.4	18.85	18.8329	0.09%
		0.7	0.2	8.92	8.9149	0.06%
			0.4	21.66	21.6648	-0.02%
ATM						
105.13	1	0.3	0.2	6.44	6.4352	0.07%
			0.4	12.9	12.8916	0.07%
		0.7	0.2	7.35	7.3463	0.05%
			0.4	14.64	14.6426	-0.02%
116.18	3	0.3	0.2	11.17	11.1614	0.08%
			0.4	22.41	22.3928	0.08%
		0.7	0.2	12.69	12.698	-0.06%
			0.4	25.12	25.1277	-0.03%
10% ITM						
94.61	1	0.3	0.2	12.37	12.3706	0.001%
			0.4	17.88	17.8846	-0.03%
		0.7	0.2	13.08	13.0876	-0.06%
			0.4	19.47	19.4697	0.001%
104.57	3	0.3	0.2	16.34	16.333	0.04%
			0.4	26.62	26.625	-0.02%
		0.7	0.2	17.71	17.7028	0.04%
			0.4	29.19	29.1794	0.04%

Table 2: Approximations and simulations: $r = 5\%, w_1 = 0.3, w_2 = 0.7, X_1(0) = 130, X_2(0) = 70$

K	T	ρ	σ	$C^{mc}[K]$	$\bar{C}[K]$	$\epsilon[K]$
10%OTM						
101.76	1	0.3	0.2	2.57	2.5694	0.02%
			0.4	8.07	8.0573	0.16%
		0.7	0.2	3.28	3.2802	-0.01%
			0.4	9.61	9.5893	0.22%
112.47	3	0.3	0.2	6.53	6.5289	0.02%
			0.4	16.65	16.6277	0.13%
		0.7	0.2	7.84	7.856	-0.20%
			0.4	19.07	19.0865	-0.09%
ATM						
92.51	1	0.3	0.2	5.69	5.6841	0.10%
			0.4	11.39	11.3822	0.07%
		0.7	0.2	6.47	6.474	-0.06%
			0.4	12.9	12.9013	-0.01%
102.24	3	0.3	0.2	9.89	9.8532	0.37%
			0.4	19.76	19.7567	0.02%
		0.7	0.2	11.18	11.1859	-0.05%
			0.4	22.12	22.1337	-0.06%
10%ITM						
83.26	1	0.3	0.2	10.9	10.8978	0.02%
			0.4	15.78	15.767	0.08%
		0.7	0.2	11.52	11.5208	-0.01%
			0.4	17.13	17.1442	-0.08%
92.02	3	0.3	0.2	14.41	14.4003	0.07%
			0.4	23.46	23.4755	-0.07%
		0.7	0.2	15.58	15.59	-0.06%
			0.4	25.69	25.6985	-0.03%

Table 3: Approximations and simulations: $r = 0\%, n = 50, T = 1$ year, $w_i = 1/50, X_i(0) = 100$

		$\sigma = 10\%$			$\sigma = 20\%$			$\sigma = 30\%$		
ρ	K	$C^{mc}[K]$	$\bar{C}[K]$	$\epsilon[K]$	$C^{mc}[K]$	$\bar{C}[K]$	$\epsilon[K]$	$C^{mc}[K]$	$\bar{C}[K]$	$\epsilon[K]$
0.3	90	10.0619	10.0619	0.0002%	10.9914	10.9913	0.0011%	12.576	12.5727	0.0261%
	95	5.534	5.5332	0.0143%	7.309	7.3058	0.0440%	9.3338	9.3351	-0.0143%
	100	2.2357	2.2353	0.0165%	4.4696	4.4696	-0.0008%	6.702	6.7019	0.0019%
	105	0.6089	0.6083	0.1055%	2.5103	2.5081	0.0875%	4.6589	4.6558	0.0674%
	110	0.1072	0.1073	-0.0702%	1.2919	1.2912	0.0530%	3.1343	3.1342	0.0046%
0.5	90	10.2095	10.2099	-0.0040%	11.8127	11.8132	-0.0039%	14.0585	14.0615	-0.0210%
	95	5.9625	5.9638	-0.0226%	8.4048	8.4047	0.0007%	11.057	11.0552	0.0164%
	100	2.8499	2.8485	0.0502%	5.6956	5.6936	0.0343%	8.5307	8.53225	-0.0182%
	105	1.075	1.0753	-0.0277%	3.6718	3.6725	-0.0195%	6.4674	6.47038	-0.0460%
	110	0.3163	0.3166	-0.0820%	2.2597	2.2586	0.0506%	4.8266	4.82663	-0.0007%
0.7	90	10.4047	10.3997	0.0480%	12.5675	12.5682	-0.0058%	15.3413	15.3437	-0.0158%
	95	6.3573	6.3574	-0.0014%	9.3254	9.3300	-0.0492%	12.4832	12.48	0.0255%
	100	3.3503	3.3511	-0.0238%	6.6988	6.6964	0.0358%	10.0361	10.0301	0.0594%
	105	1.498	1.4977	0.0171%	4.653	4.6503	0.0574%	7.9749	7.9718	0.0389%
	110	0.5646	0.5645	0.0227%	3.1318	3.1291	0.0875%	6.265	6.27094	-0.0948%

5 Conclusion

This paper handles the problem of pricing index options. The index option price is influenced by the distribution of the individual components and the dependence structure, which makes it a hard task to derive closed form solutions. We assume that the risk-neutral dynamics of the stocks can be described by a multivariate Black & Scholes model. In this simple stock price model, the vanilla options can be priced using the celebrated Black & Scholes option pricing formula, but the price $C[K]$ of an index option is not given in an analytical formula. We derive a closed form approximation, which is based on convex approximations for a sum S of dependent lognormal random variables. Comparing our new approximate option pricing formula with Monte Carlo simulations shows that the approximate values are close to the simulated values. Furthermore, we also show that the approximate index option curve $\overline{C}$ can be interpreted as the index option curve under the approximate index $\overline{S}$, where the cdf of $\overline{S}$ has a more attractive form than the cdf of the original index S.

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