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On the interplay between distortion, mean value and Haezendonck–Goovaerts risk measures[☆]

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ABSTRACT

In the actuarial research, distortion, mean value and Haezendonck–Goovaerts risk measures are concepts that are usually treated separately. In this paper we indicate and characterize the relation between these different risk measures, as well as their relation to convex risk measures. While it is known that the mean value principle can be used to generate premium calculation principles, we will show how they also allow to generate solvency calculation principles. Moreover, we explain the role provided for the distortion risk measures as an extension of the Tail Value-at-Risk (TVaR) and Conditional Tail Expectation (CTE).

1. Introduction

A risk measure is a mapping from a class of random variables to the real numbers. A variety of risk measures is presented and discussed in the actuarial literature and the choice of the risk measure depends on the area in which it will be used. For example, risk measures used for premium calculations differ from the ones used for setting capital requirements and reserving. In the sequel, we will consider the random variable X, representing the loss an insurance company is facing. Its distribution function is denoted by F_X. The corresponding risk measure is denoted by $H(X)$. So $H(X)$ can represent the premium, charged by the insurance company to insure the risk X, or it can represent the solvency capital a company has to hold for being exposed to the risk X.

In this paper we will investigate three classes of risk measures: the distortion risk measures, the mean value principles and the Haezendonck–Goovaerts risk measures. We will show that a mean value principle can be used to define the Haezendonck–Goovaerts risk measure. Furthermore, we will use a distortion function to construct a more general version of the Haezendonck–Goovaerts risk measure, called the generalized Haezendonck–Goovaerts risk measure.

Throughout the paper, we use $\mathcal{B}$ to denote the class of all bounded risks and we only consider risks that belong to $\mathcal{B}$. Moreover, we will always silently assume that all the risks are non-negative.

2. Mean value risk measures

Mean value risk measures have been extensively studied in the framework of insurance premiums; see e.g. Goovaerts et al. (1984). A beautiful characterization of the mean value principle, based on the iterativity property, has been given by Gerber (1974); see also Gerber (1979). To derive this characterization, the following continuity condition was introduced.

Definition 2.1 (*Continuity Condition*)**.** Consider the random variable X_{aq} which represents a risk:

$$\Pr\left(X_{aq} = a\right) = q, \qquad (2.1)$$
$$\Pr\left(X_{aq} = 0\right) = 1 - q.$$

For a fixed $a > 0$, the risk measure H satisfies the *continuity condition* if, and only if, $H(X_{aq})$ is strictly increasing for $0 \leq q \leq 1$, with $H(X_{a0}) = 0$ and $H(X_{a1}) = a$.

A risk measure H is said to satisfy the iterativity property if

$$H(X) = H\left(H\left(X \mid Y\right)\right),$$

[☆] The referees as well as the authors of Tang and Yang (2012) and Bellini and Rosazza Gianin (2011), have strongly insisted on the name Haezendonck–Goovaerts risk measure as was discussed during the 15th IME congress in Trieste.

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where $X \mid Y$ is the conditional random variable X, given Y. Next, we recall the definition of the mean value principle.

Definition 2.2. A risk measure H is said to be generated by the *mean value principle* if there exists a strictly increasing function v such that $v(H(X)) = \mathrm{E}[v(X)]$.

Gerber (1974) proved the following theorem, changing the notion of premium calculation into risk measures.

Theorem 2.1. *A risk measure H satisfying the continuity condition of Definition 2.1 is iterative if, and only if, it is generated by the mean value principle.*

Based on risk measures one can consider many types of problems, four of which are explained in the following paragraphs.

2.1. Comparing risk measures

Comparison of different risk measures is possible by changing the underlying function v for the mean value principle. This gives rise to the notion of comparability of risk measures.

Definition 2.3 (*Comparable Mean Value Risk Measures*)**.** Two mean value principles H, with strictly increasing v_1 and v_2, are comparable if for all bounded risks X

$$H\left(X, v_1\right) \leq H\left(X, v_2\right),$$

or the reverse inequality, with $H\left(X, v\right) = v^{-1}\left(\mathrm{E}\left[v(X)\right]\right)$.

The following theorem is proven in Goovaerts et al. (1984):

Theorem 2.2. *Let v_1 and v_2 be two continuous and strictly increasing functions in $\mathbb{R}$. A necessary and sufficient condition for $H\left(X, v_1\right)$ and $H\left(X, v_2\right)$ to be comparable is that the function*

$$f = v_2 v_1^{-1}$$

satisfies

$$f\left(\mathrm{E}\left[X\right]\right) \leq \mathrm{E}\left[f(X)\right],$$

or the reverse inequality, for all bounded risks X. Hence, f has to be a convex or a concave function.

2.2. Risk ordering

A second application of risk measures is risk ordering, where the same principle is used to calculate the risk of different random variables or prospects. Consider a mean value principle and compare the "risk premiums" $H(X)$ and $H(Y)$, where $v\left(H(X)\right) = \mathrm{E}\left[v(X)\right]$ and $v\left(H(Y)\right) = \mathrm{E}\left[v(Y)\right]$. Consider two risks X and Y such that Y is larger than X in the stochastic dominance, we write $X \leq_1 Y$. In case v is increasing, we have that $\mathrm{E}\left[v(X)\right] \leq \mathrm{E}\left[v(Y)\right]$ and as a consequence we also find that $H(X) \leq H(Y)$.

In case X and Y are ordered according to stop-loss order, i.e. $\mathrm{E}\left[(X-t)_+\right] \leq \mathrm{E}\left[(Y-t)_+\right]$ for all t, we have that $\mathrm{E}\left[v(X)\right] \leq \mathrm{E}\left[v(Y)\right]$ for all non-decreasing convex functions v, and consequently $H(X) \leq H(Y)$. Stop-loss order is also referred to as second order stochastic dominance.

A third notion is convex order, where one assumes that

$$\mathrm{E}\left[(X-t)_+\right] \leq \mathrm{E}\left[(Y-t)_+\right] \quad \text{for all } t,$$
$$\text{and} \quad \mathrm{E}\left[X\right] = \mathrm{E}[Y].$$

In Dhaene et al. (2002a) it is shown that these conditions are equivalent to

$$\mathrm{E}\left[(t-X)_+\right] \leq \mathrm{E}\left[(t-Y)_+\right] \quad \text{for all } t,$$
$$\text{and} \quad \mathrm{E}\left[X\right] = \mathrm{E}[Y].$$

In this case, $\mathrm{E}\left[v(-X)\right] \leq \mathrm{E}\left[v(-Y)\right]$ for all convex functions v, where $\mathrm{E}\left[X\right] = \mathrm{E}[Y]$. This means that the tails of Y are fatter than the ones of X. Considering the calculation of premiums by means of mean value principles, larger premiums for stop-loss ordered random variables are obtained, or, in the framework of reinsurance, higher reinsurance premiums. Bounds for stop-loss expressions under several constraints are given in the literature; see e.g. Goovaerts et al. (2011). A recent reference about risk measures and risk ordering is Song and Yan (2009).

2.3. Optimal risk measures

A third application of risk measures is the possibility to generate risk measures which are "optimal". Considering a mean value principle with an exponential function $v(x) = \mathrm{e}^{\alpha x}$ one gets the exponential premium calculation principle:

$$H(X) = \frac{1}{\alpha} \ln \mathrm{E}\left[\mathrm{e}^{\alpha X}\right].$$

On the other hand, one has the following set of inequalities

$$\begin{aligned} \frac{1}{\alpha} \ln \mathrm{E}\left[\mathrm{e}^{\alpha X}\right] &= \frac{1}{\alpha} \int_0^{\alpha} \frac{\mathrm{E}\left[\mathrm{e}^{sX} X\right]}{\mathrm{E}\left[\mathrm{e}^{sX}\right]} \mathrm{d}s \\ &\leq \frac{\mathrm{E}\left[\mathrm{e}^{\alpha X} X\right]}{\mathrm{E}\left[\mathrm{e}^{\alpha X}\right]}, \end{aligned}$$

which gives rise to the Esscher premium.

Weighted Esscher premiums have been characterized using a set of axioms; see Goovaerts et al. (2004), leading to

$$H(X) = \int_{-\infty}^{+\infty} \frac{\mathrm{E}\left[\mathrm{e}^{tX} X\right]}{\mathrm{E}\left[\mathrm{e}^{tX}\right]} \mathrm{d}G(t),$$

where $G : \mathbb{R} \longrightarrow [0, 1]$ and G is concave on $(0, +\infty)$ and convex on $(-\infty, 0)$. Consequently $H(X)$ can be written as $\mathrm{E}^*\left[X\right]$, where the * indicates that the expectation is calculated using the differential

$$\mathrm{d}F_X^{G(\cdot)}(x) = \int_{-\infty}^{+\infty} \frac{\mathrm{e}^{tX} \mathrm{d}G(t)}{\mathrm{E}\left[\mathrm{e}^{tX}\right]} \mathrm{d}F_X(x). \tag{2.2}$$

For further applications of risk measures, the interested reader is referred to Goovaerts et al. (1984), Goovaerts et al. (2010a), Franco and Tankov (2011), Brahimi et al. (2011), Asimit et al. (2011), Bellini and Rosazza Gianin (2008) and Kaas et al. (2008).

2.4. Additivity properties for risk measures

Another interesting application is the possibility to examine additivity properties of risk measures. Indeed, in case X and Y are independent random variables, we might be interested in risk measures that satisfy the axiom

$$H\left(X+Y\right) = H(X) + H(Y).$$

Mixtures of Esscher premiums and mixtures of exponential premiums can be characterized; see e.g. Gerber and Goovaerts (1981). Moreover, an endogenous characterization can be derived for the additivity property of comonotonic risks; see Goovaerts et al. (2004). For more information on axiomatic characterizations of risk measures we refer to Goovaerts and Laeven (2008) and Goovaerts et al. (2010a,b).

3. Application of the mean value principle to generate distortion risk measures

Consider a random variable X, belonging to the class $\mathcal{B}$. Its distribution function is denoted by F_X. An increasing distortion risk function $g : [0, 1] \longrightarrow [0, 1]$, with $g(0) = 0$ and $g(1) = 1$, is used to calculate the *distortion risk measure* $\rho_g(X)$, for a (bounded and positive) loss X as follows:

$$\rho_g(X) = \int_0^{\infty} g\left(1 - F_X(x)\right) \mathrm{d}x, \tag{3.1}$$

or equivalently, by means of a partial integration and a substitution, as

$$\rho_g(X) = \int_0^{1} F_X^{-1}(y) g'\left(1 - y\right) \mathrm{d}y. \tag{3.2}$$

In the sequel of this paper we will always assume that g is differentiable such that (3.2) is well defined. Extending (3.2) to the situation where the random variable has positive and negative realizations is straightforward. In case $g_1(y) \geq g_2(y)$ for all y, it follows that $\rho_{g_1}(X) \geq \rho_{g_2}(X)$. As a special case, choosing $g_2(x) = x$ implies that $\rho_g(X) \geq \mathrm{E}[X]$ for any distortion risk measure ρ_g, with $g(x) \geq x$. Distortion functions were introduced in Yaari (1987) and more recently treated by Brahimi et al. (2011).

Two random variables X^c and Y^c are *comonotonic* if they are maximally dependent. Indeed, for comonotonic random variables one has that two possible outcomes (x_1, y_1) and (x_2, y_2) should always be ordered componentwise: $x_1 \leq x_2$ and $y_1 \leq y_2$ should hold or the other way around. The concept of comonotonicity can easily be extended to the general case where we deal with n random variables. In this situation, all possible outcomes should be ordered componentwise. A good reference in this context is Dhaene et al. (2002a). A whole range of financial and actuarial applications are described in Dhaene et al. (2002b). An extension of the concept of comonotonicity and convex order relations can be found in Cheung (2009) and Nam et al. (2011). Applications of (conditional) comonotonicity to Value-at-Risk-based risk management can be found in Laeven (2009).

The quantiles of a comonotonic sum $X^c + Y^c$ can be decomposed as follows:

$$F_{X^c+Y^c}^{-1}(p) = F_{X^c}^{-1}(p) + F_{Y^c}^{-1}(p), \quad p \in [0, 1].$$

Using expression (3.2), it is easy to show that distortion risk measures ρ_g are additive for comonotonic risks X^c and Y^c:

$$\rho_g(X^c + Y^c) = \rho_g(X^c) + \rho_g(Y^c).$$

For the risk X_{aq} which was defined in (2.1), we get

$$\rho_g(X_{aq}) = (1 - q)0 + \int_{1-q}^{1} a g'(1 - y)\mathrm{d}y = a g(q),$$

which implies that any distortion risk measure satisfies the continuity condition of Definition 2.1. As a next step, we can examine whether distortion risk measures can be used to define an appropriate mean value risk measure. If we add the property of iterativity we are within the context of Theorem 2.1 and there exists a function v such that for X_{aq} the following holds:

$$v\left(\rho_g\left(X_{aq}\right)\right) = \mathrm{E}\left[v\left(X_{aq}\right)\right],$$

which is equivalent with:

$$v(a g(q)) = q v(a). \tag{3.3}$$

It is known that using v to calculate the mean value risk measure does not give translation invariant risk measures, nor positive homogeneity.

Taking the derivative with respect to a on both sides of Eq. (3.3) and putting $a = 0$, we find that the function g is given by

$$g(q) = q. \tag{3.4}$$

Combining (3.3) and (3.4) it follows that

$$v'(aq) = v'(a),$$

implying that both v and g have to be linear functions, reducing the mean value principle and the distortion risk measure to a pure expectation principle. So it follows that the set of iterative distortion risk measures only contains the expectation principle. In the next section we will show that it is possible to link a mean value risk measure to a distortion function. This might give the impression that mean value principles cannot produce positive homogeneous and translation invariant risk measures. We will solve this shortcoming by introducing a new random variable Z.

4. Application of a mean value principle to generate distortion risk measures in the framework of capital requirements

We have seen that positive homogeneous or translation invariant principles cannot be obtained when using a mean value principle to generate distortion risk measures. In the light of capital requirements, a good risk measure should focus more on the risk in the tails than on the total risk. Indeed, when setting an additional capital for potential heavy losses, one tries to avoid the situation where potential gains and potential (heavy) losses will balance each other out. For a bounded risk X, a risk capital $\rho(X)$ is determined using a transformed random variable Z, which only has probability mass in the right tail.

This section will show how to choose the variable Z, such that using a mean value principle to determine the risk Z can be linked with a distortion risk measure and a Haezendonck–Goovaerts risk measure for the initial risk X.

4.1. *Risk measures for capital requirements*

Consider the random variable Z, which is defined as

$$Z = \frac{\left(F_X^{-1}(U) - t\right)_+}{\rho - t},$$

with U uniformly distributed on the unit interval and $\rho, t \in \mathbb{R}^+$. A broad literature exists concerning the derivation of bounds on expectations; see e.g: Goovaerts et al. (2010a, 2011). Assume we have a (risk) capital t available. The random variable Z can be used to determine and measure the capital to be considered for the residual risk. If ρ represents the sum of the risk capital and a measure of the residual risk (above t) it is clear that $\rho - t \geq 0$. Hence, Z can be interpreted as the random variable that gives the ratio between the residual risk $\left(F_X^{-1}(U) - t\right)_+$ and the supplementary needed capital. We could apply the mean value risk measure for some valid function v and obtain

$$v\left(H(Z)\right) = \mathrm{E}\left[v(Z)\right].$$

Only for those realizations of Z for which $Z > 1$ a residual risk arises. For $Z < 1$, residual gains are obtained. For the moment, we take $v(x) = x$ for all x. If $H(Z)$ is forced to be equal to $1-\alpha$, for some $\alpha \in (0, 1)$, the risk capital is denoted by $\rho_l(X, t)$. Here we use the subscript l because a linear function v is used. Given the available capital t, $\rho_l(X, t)$ can be derived from the following equation:

$$H(Z) = \int_0^1 \frac{\left(F_X^{-1}(u) - t\right)_+}{\rho_l(X, t) - t} \mathrm{d}u = 1 - \alpha. \tag{4.1}$$

Rewriting (4.1) leads to:

$$\mathrm{E}\left[\frac{\left(F_X^{-1}(U) - t\right)_+}{\rho_l(X, t) - t}\right] = 1 - \alpha, \tag{4.2}$$

from which it follows that the risk capital $\rho_l(X, t)$ is given by:

$$\begin{aligned} \rho_l(X, t) &= t + \frac{1}{1 - \alpha} \int_t^{+\infty} (x - t)_+ \mathrm{d}F_X(x) \\ &= t + \int_t^{+\infty} \frac{1 - F_X(x)}{1 - \alpha} \mathrm{d}x. \end{aligned} \tag{4.3}$$

Hence, $\rho_l(X,t) - t$ seems to be expressed as a distortion risk measure. However, the corresponding function $g(x) = \frac{x}{1-\alpha}$ is not a distortion function because $g(1) = \frac{1}{1-\alpha} > 1$.

We can "solve" this by taking $t = F_X^{-1}(\alpha)$ and defining the function g as:

$$g(x) = \min\left\{\frac{x}{1-\alpha}, 1\right\}. \tag{4.4}$$

The function g is now a valid distortion function. From (3.1), we find that the distortion risk measure $\rho_g\left(X, F_X^{-1}(\alpha)\right)$ is equal to:

$$\begin{aligned}\rho_g(X, F_X^{-1}(\alpha)) &= \int_0^{+\infty} g\left(1 - F_X(x)\right) \mathrm{d}x \\ &= \int_0^{F_X^{-1}(\alpha)} 1\mathrm{d}x + \int_{F_X^{-1}(\alpha)}^{+\infty} \frac{1-F_X(x)}{1-\alpha}\mathrm{d}x \\ &= t + \int_t^{+\infty} \frac{1-F_X(x)}{1-\alpha}\mathrm{d}x.\end{aligned}$$

In other words:

$$\rho_g(X, F_X^{-1}(\alpha)) = \rho_l\left(X, F_X^{-1}(\alpha)\right). \tag{4.5}$$

Clearly, starting from a mean value principle applied on the transformed random variable Z, the risk capital $\rho_l\left(X, F_X^{-1}(\alpha)\right)$ can be linked with the risk capital $\rho_g(X, F_X^{-1}(\alpha))$. Here, $\rho_l\left(X, F_X^{-1}(\alpha)\right)$ is calculated using a mean value principle whereas $\rho_g\left(X, F_X^{-1}(\alpha)\right)$ is derived with the help of a distortion function. Note that this relation between mean value principles and distortion risk measures only holds for the special choice $t = F_X^{-1}(\alpha)$.

In general, there is no need to take $t = F_X^{-1}(\alpha)$. Hereafter, we will show that the choice $t = F_X^{-1}(\alpha)$ is optimal in the sense that it results in the minimal value for $\rho_l(X,t)$. For this optimal choice of t, we will omit the dependence on t in the notation. For example, we write $\rho_l(X)$ instead of $\rho_l\left(X, F_X^{-1}(\alpha)\right)$.

Consider two comonotonic random variables X_1^c and X_2^c and take $X = X_1^c + X_2^c$. Choosing $t = F_X^{-1}(\beta)$, with $\beta < 1$, it can be seen that

$$\begin{aligned}&\rho_l\left(X_1^c + X_2^c\right) - F_{X_1^c}^{-1}(\beta) - F_{X_2^c}^{-1}(\beta) \\ &\quad = \rho_l(X_1) - F_{X_1}^{-1}(\beta) + \rho_l(X_2) - F_{X_2}^{-1}(\beta),\end{aligned}$$

or

$$\begin{aligned}&\rho_l\left(\left(X_1^c + X_2^c - F_{X_1^c+X_2^c}^{-1}(\beta)\right)_+\right) \\ &\quad = \rho_l\left(\left(X_1^c - F_{X_1^c}^{-1}(\beta)\right)_+\right) + \rho_l\left(\left(X_2^c - F_{X_2^c}^{-1}(\beta)\right)_+\right),\end{aligned}$$

because for comonotonic risks, the following holds automatically for n random variables:

1. the quantiles are additive: $F_{X_1^c+\cdots+X_n^c}^{-1}(\beta) = \sum_{i=1}^n F_{X_i^c}^{-1}(\beta)$, and
2. the right tail of the comonotonic sum can be decomposed into a sum of random variables, describing the riskiness in the right tails:

$$\left(X_1^c + \cdots + X_n^c - F_{X_1^c+\cdots+X_n^c}^{-1}(\beta)\right)_+ = \sum_{i=1}^n \left(X_i^c - F_{X_i^c}^{-1}(\beta)\right)_+,$$

provided the random variables are continuous.

Denote the residual risk measure, given the capital t, as $\pi(X,t) = \rho - t$. Then $\pi(X^c,t)$ for $X^c = X_1^c + X_2^c + \cdots + X_n^c$ is determined as (in case $v(x) = x$):

$$1-\alpha = \int_\beta^1 \frac{\sum_j \left(F_{X_j^c}^{-1}(u) - F_{X_j^c}^{-1}(\beta)\right)_+}{\pi_l(X^c,t)}\mathrm{d}u,$$

where $\pi_l(X^c,t)$ is obtained by means of a particular mean value principle with a linear function v.

As a conclusion we see that the risk measure $\rho_l(X,t)$ in (4.2), which is derived out of a mean value principle, is comonotone additive. For the special case where $t = F_X^{-1}(\alpha)$, we have seen that the risk measure can be expressed as a distortion risk measure. In what follows, the role of the Haezendonck–Goovaerts risk measure will be explained in terms of discrete distribution functions.

4.2. *Haezendonck–Goovaerts risk measure*

The Haezendonck–Goovaerts risk measure was first introduced by Haezendonck and Goovaerts (1982) and called the Haezendonck risk measure in Goovaerts et al. (2004). During the 15th International IME congress in Trieste, some participants agreed that it would be more proper to call it the Haezendonck–Goovaerts risk measure. Therefore, we will also use this name to be consistent with a few other papers recently emerging in this area. Recent references are Bellini and Rosazza Gianin (2008), Nam et al. (2011) and Kratschmer and Zahle (2011).

We start this section with the definition of the Haezendonck–Goovaerts risk measure. Consider a random variable X, satisfying $-\infty < \min[X] \le \max[X] < +\infty$ and a function φ, which is strictly increasing and has $\varphi(0) = 0, \varphi(1) = 1$ and $\varphi(+\infty) = +\infty$. For any $t \in \mathbb{R}$ and $\rho > t$, we have from Goovaerts et al. (2003):

$$\Pr(X > \rho) = \Pr(X - t > \rho - t) \le \mathrm{E}\left[\varphi\left(\frac{(X-t)_+}{\rho - t}\right)\right].$$

It can be shown that the equation

$$\mathrm{E}\left[\varphi\left(\frac{(X-t)_+}{\rho - t}\right)\right] = 1 - \alpha, \tag{4.6}$$

with $\alpha \in (0,1)$, always has a solution, as long as $-\infty < t < \max[X]$. This solution is denoted by $\rho_\varphi(X,t)$ and determines the residual risk $(\rho - t)$ of the tail $(X-t)_+$. Furthermore, one has that $\rho_\varphi(X,t) > F_X^{-1}(\alpha)$. Hereafter, we denote the solution of (4.6) obtained using a linear function φ by ρ_l, whereas the more general case is denoted by ρ_φ. The *Haezendonck–Goovaerts risk measure* is defined as follows:

Definition 4.1. Let φ be a strictly increasing function with $\varphi(0) = 0, \varphi(1) = 1, \varphi(+\infty) = +\infty$ and let $\alpha \in (0,1)$. The *Haezendonck–Goovaerts risk measure* is denoted by $\rho_\varphi(X)$ and equals:

$$\rho_\varphi(X) = \inf_{-\infty < t < \max[X]} \rho_\varphi(X,t),$$

where $\rho_\varphi(X,t)$ is the solution of Eq. (4.6).

4.2.1. *Properties*

In case we calculate the Haezendonck–Goovaerts risk measure as follows, putting $t = F_X^{-1}(\beta)$:

$$1-\alpha = \mathrm{E}\left[\varphi\left(\frac{\left(F_X^{-1}(U) - F_X^{-1}(\beta)\right)_+}{\rho_\varphi(X,t) - F_X^{-1}(\beta)}\right)\right], \tag{4.7}$$

the function $\rho_\varphi(X,t)$ is positive homogeneous and translation invariant. Indeed, we immediately see that:

$$\begin{aligned}&\rho_\varphi(aX,t) = a\rho_\varphi(X,t), \quad \text{for } a > 0, \\ &\rho_\varphi(a+X,t) = a + \rho_\varphi(X,t) \quad \text{for } a \in \mathbb{R}.\end{aligned}$$

Therefore, the Haezendonck–Goovaerts risk measure is also homogeneous and translation invariant. A proof of this result is given already in Goovaerts et al. (2004), where it is also shown that a convex function φ gives subadditivity for the pair (X, Y), provided

$$\max[X+Y] = \max[X] + \max[Y]. \tag{4.8}$$

In the construction of the Haezendonck–Goovaerts risk measure, t or $F_X^{-1}(\beta)$ has to be specified in an optimal way. Let us first consider the choice $\varphi(x) = x$. For this special case, we obtain

$$\mathrm{E}\left[\frac{(X-t)_+}{\rho - t}\right] = 1-\alpha,$$

and, consequently,

$$\rho = t + \frac{1}{1-\alpha}\mathrm{E}\left[(X-t)_+\right].$$

The Haezendonck–Goovaerts risk measure arises if t is selected such that ρ becomes minimal.

Theorem 4.1. [1] *Consider the function φ which is assumed to be differentiable. The Haezendonck–Goovaerts risk measure $\rho_\varphi(X)$ is determined as the solution of the system of equations:*

$$1-\alpha = \int_t^{+\infty} \varphi\left(\frac{(x-t)_+}{\rho - t}\right) \mathrm{d}F_X(x), \tag{4.9}$$

$$\rho = t + \frac{\int_t^{+\infty} \varphi'\left(\frac{(x-t)_+}{\rho-t}\right)(x-t)_+ \,\mathrm{d}F_X(x)}{\int_t^{+\infty} \varphi'\left(\frac{(x-t)_+}{\rho-t}\right) \mathrm{d}F_X(x)}. \tag{4.10}$$

Proof. This result follows directly by taking the derivative in (4.9) and putting $\rho'(t) = 0$. □

Special cases are:

1. $\varphi(x) = x$:

$$\rho = F_X^{-1}(\alpha) + \frac{1}{1-\alpha}\mathrm{E}\left[\left(X - F_X^{-1}(\alpha)\right)_+\right].$$

2. $\varphi(x) = \mathrm{e}^{\alpha x}$:

$$\rho = t + \frac{\mathrm{E}\left[(X-t)_+ \mathrm{e}^{\frac{\alpha X}{\rho - t}}\right]}{\mathrm{E}\left[\mathrm{e}^{\frac{\alpha X}{\rho-t}}\right]}.$$

Example 4.1 (*VAR*)**.** Consider the r.v. S. For a fixed $\alpha \in (0, 1)$, $\rho_l(S, t)$, given by:

$$\rho_l(S,t) = t + \frac{1}{1-\alpha}\mathrm{E}\left[(S-t)_+\right], \tag{4.11}$$

is the solution of the equation:

$$\mathrm{E}\left[\frac{(S-t)_+}{\rho_l(S,t) - t}\right] = 1-\alpha.$$

We have that $\mathrm{TVaR}(S,\alpha) = \rho_l(S, \mathrm{VaR}[S,\alpha])$. When we are taking the comparability of the mean value principles, one has immediately that the equation

$$\mathrm{E}\left[\varphi\left(\frac{(S-t)_+}{\rho_\varphi(S,t) - t}\right)\right] = 1-\alpha, \tag{4.12}$$

gives

$$\rho_\varphi(S,t) > \rho_l(S,t).$$

So we find that

$$\mathrm{TVaR}(S,\alpha) \le \rho_\varphi(S, \mathrm{VaR}[S,\alpha]). \tag{4.13}$$

[1] This result has been presented by the authors at the meeting of the "Deutsche Gesellschaft für Versicherungs und Finanzmathematik" (DGVFM), in April 2009.

Example 4.2 (*Haezendonck–Goovaerts Risk Measure for Two Point Distributions*)**.** In Example 3.1 of Goovaerts et al. (2004), a Bernoulli random variable B_q is considered, with $\Pr(B_q = 1) = 1 - \Pr(B_q = 0) = q$. Using the function $\varphi(x) = x$, the optimal Haezendonck–Goovaerts risk measure is denoted by $\rho_l(B_q)$ and equal to:

$$\rho_l(B_q) = \min\left\{\frac{q}{1-\alpha}, 1\right\}.$$

Proceeding along the same lines, the following equation is obtained for a general choice of $\varphi(x)$:

$$\rho_\varphi(B_q) = \min\left\{\frac{1}{\varphi^{-1}\left(\frac{1-\alpha}{q}\right)}, 1\right\}. \tag{4.14}$$

Next, we consider the distribution $\left(aB_q - t\right)_+$, where $a > t$:

$$\Pr\left(aB_q - t = a - t\right) = 1 - \Pr\left(aB_q - t = 0\right) = q.$$

We get:

$$\rho_l\left(\left(aB_q - t\right)_+\right) = (a-t)\min\left\{\frac{q}{1-\alpha}, 1\right\},$$

and also

$$\rho_\varphi\left(\left(aB_q - t\right)_+\right) = (a-t)\min\left\{\frac{1}{\varphi^{-1}\left(\frac{1-\alpha}{q}\right)}, 1\right\}. \tag{4.15}$$

4.2.2. The connection between the Haezendonck–Goovaerts risk measure and distortion risk measures

The previous example, describing the Haezendonck–Goovaerts risk measure for two point distribution, gives rise to the following question: given a distortion function g and the corresponding risk measure for a Bernoulli random variable/risk B_q, can one determine the corresponding function φ of the Haezendonck–Goovaerts risk measure to find back the distortion risk measure.

Theorem 4.2. *Consider a Bernoulli risk B_q and a function $g(q)$ which is a distortion measure function such that $g(q)$ is increasing for $0 < q < 1-\alpha$ and $g(q) = 1$ for $1-\alpha < q \le 1$. A sufficient condition for the existence of a function φ for which the equality $\rho_\varphi(B_q) = \rho_g(B_q)$ holds, is that $g(q)$ is concave for $q \le 1-\alpha$.*

Proof. For a Bernoulli risk B_q, the equality $\rho_\varphi\left(B_q\right) = \rho_g\left(B_q\right)$ induces the equality

$$\varphi\left(\frac{1}{g(q)}\right) = \frac{1-\alpha}{q}.$$

Let $c(q) = \frac{1}{g(q)}$, then

$$c'(q) = -\frac{g'(q)}{g^2(q)} \le 0,$$

$$c''(q) = -\frac{g''(q)\,g(q) - 2\left(g'(q)\right)^2}{g^3(q)} \ge 0,$$

for a concave distortion function g. Hence, $c(q)$ is a decreasing convex function for $0 < q < 1-\alpha$. If we set $x = c(q)$ (or $q = c^{-1}(x)$), we have that

$$\frac{\mathrm{d}x}{\mathrm{d}q} = c'(q),$$

which also means that

$$\frac{\mathrm{d}q}{\mathrm{d}x} = \frac{1}{c'(q)} = \frac{1}{c'\left(c^{-1}(x)\right)} < 0.$$

For the second derivative we find

$$\frac{\mathrm{d}^2 q}{\mathrm{d}x^2} = -\frac{c''(q)}{\left(c'(q)\right)^3} \geq 0.$$

Consequently, $c^{-1}(x)$ is a decreasing and convex function. The function $\varphi(x)$ can be expressed in terms of $c^{-1}(x)$ as follows:

$$\varphi(x) = \frac{1-\alpha}{c^{-1}(x)},$$

and

$$\varphi'(x) = -(1-\alpha)\frac{\left(c^{-1}(x)\right)'}{\left(c^{-1}(x)\right)^2}.$$

The second derivative can be written as:

$$\varphi''(x) = -(1-\alpha)\frac{\left(c^{-1}(x)\right)'' c^{-1}(x) - 2c^{-1}(x)'^2}{\left(c^{-1}(x)\right)^3}.$$

Using the function $c(q)$, we can write

$$\varphi''(x) = \frac{-(1-\alpha)}{q^3}\left[-\frac{c''(q)}{c'(q)^3}q - \frac{2}{c'(q)^2}\right]$$

and directly in terms of the distortion function as

$$\varphi''(x) = \frac{-(1-\alpha)}{q^3}\left[-\frac{g''(q)}{g(q)^2}q + \frac{2g'(q)^2}{g(q)^3} + 2\frac{g(q)^4}{g'(q)}\right].$$

As soon as g is increasing and concave, φ must be convex. □

Let us recall the following result from Wirch and Hardy (2000) and Dhaene and Goovaerts (1998):

Theorem 4.3. *The distortion risk measure $\rho_g(X) = \int_0^{+\infty} g\left(1 - F_X(x)\right)\mathrm{d}x$ is subadditive if, and only if, g is a concave distortion function.*

We can formulate the following theorem:

Theorem 4.4. *A Haezendonck–Goovaerts risk measure, with φ derived from a concave distortion function g is subadditive.*

Proof. In Goovaerts et al. (2004), it is shown that a convex φ results in a Haezendonck–Goovaerts risk measure which is subadditive. Combining the results of Theorems 4.2 and 4.3 results in the fact that the Haezendonck–Goovaerts risk measure derived from a concave distortion function g has to produce subadditive risk measures. □

4.3. The generalized Haezendonck–Goovaerts risk measure

Assume a capital $\rho_l\left(X, F_X^{-1}(\alpha)\right)$ is available for a risk X. To ease the notation, we will write ρ_l instead of $\rho_l\left(X, F_X^{-1}(\alpha)\right)$. In the previous paragraph, we derived in (4.3) the residual risk

$$\pi_l\left(X, F_X^{-1}(\alpha)\right) = \rho_l - F_X^{-1}(\alpha) = \frac{1}{1-\alpha}\mathrm{E}\left[\left(F_X^{-1}(U) - F_X^{-1}(\alpha)\right)_+\right]$$

as the solution of

$$\mathrm{E}\left[\frac{\left(F_X^{-1}(U) - F_X^{-1}(\alpha)\right)_+}{\rho_l - F_X^{-1}(\alpha)}\right] = 1-\alpha. \tag{4.16}$$

A generalization is possible for arbitrary distortion risk measures in the following way. Let

$$\rho_g(X, F_X^{-1}(\alpha)) = \int_0^1 F_X^{-1}(y)g'(1-y)\mathrm{d}y,$$

where g is defined as in (4.14) and equal to $g(x) = \min\left\{\frac{x}{1-\alpha}, 1\right\}$. Consider the following relation:

$$\mathrm{E}\left[\frac{\left(F_X^{-1}(U) - F_X^{-1}(\alpha)\right)_+ g'(1-U)}{\rho_{l,g} - F_X^{-1}(\alpha)}\right] = 1-\alpha. \tag{4.17}$$

Solving for $\rho_{l,g}$ gives

$$\begin{aligned}\rho_{l,g} - F_X^{-1}(\alpha) &= \frac{1}{1-\alpha}\mathrm{E}\left[\left(F_X^{-1}(U) - F_X^{-1}(\alpha)\right)_+ g'(1-U)\right] \\ &= \frac{1}{1-\alpha}\int_\alpha^1 \left(F_X^{-1}(u) - F_X^{-1}(\alpha)\right) g'(1-u)\,\mathrm{d}u \\ &= \frac{1}{1-\alpha}\left(\int_\alpha^1 F_X^{-1}(u)g'(1-u)\,\mathrm{d}u \right.\\ &\quad \left. - \int_\alpha^1 F_X^{-1}(\alpha)\,g'(1-u)\,\mathrm{d}u\right).\end{aligned}$$

Using the fact that $g'(1-u) = 0$, for $u < \alpha$ and the relations $g(0) = 0, g(1-\alpha) = 1$, we find

$$\begin{aligned}\rho_{l,g} &= F_X^{-1}(\alpha) + \frac{1}{1-\alpha}\int_0^1 F_X^{-1}(u)g'(1-u)\,\mathrm{d}u - \frac{F_X^{-1}(\alpha)}{1-\alpha} \\ &= F_X^{-1}(\alpha) + \frac{1}{1-\alpha}\left(\rho_g\left(X, F_X^{-1}(\alpha)\right) - F_X^{-1}(\alpha)\right).\end{aligned} \tag{4.18}$$

Expression (4.18) shows that a distortion function can be used to generate risk measures, which are solutions of Eq. (4.17). This solution can be linked with the solution of Eq. (4.16), using relation (4.5).

Because $g' > 0$ and $\mathrm{E}[g'(1-U)] = 1, g'(1-u)$ can be considered as a density in $u \in (\alpha, 1)$. As a consequence we can apply a distortion to the distribution directly within the calculation of the Haezendonck–Goovaerts risk measure, using Theorem 2.2 to generate for convex φ. Hereafter, the distortion function used in this generalized version of the Haezendonck–Goovaerts risk measure will be denoted by h.

Definition 4.2 (*Generalized Haezendonck–Goovaerts Risk Measure*)**.** Let φ be a strictly increasing function with $\varphi(0) = 0, \varphi(1) = 1, \varphi(+\infty) = +\infty$. Let $\alpha \in (0, 1)$ and $-\infty < t < F_X^{-1}(U) = \max(X)$. Given a distortion function h, which is an increasing function, satisfying $h(0) = 0$ and $h(1) = 1$, the *generalized Haezendonck–Goovaerts risk measure* is denoted by $\rho_{\varphi,h}(X, t)$ and is the solution of:

$$1-\alpha = \mathrm{E}\left[\varphi\left(\frac{\left(F_X^{-1}(U) - t\right)_+}{\rho_{\varphi,h}(X,t) - t}\right)h'(1-U)\right], \tag{4.19}$$

where $U \sim$ Uniform (0, 1).

If t is selected such that this generalized Haezendonck–Goovaerts risk measure is minimal, we find the so called *optimal generalized Haezendonck–Goovaerts risk measure*. The proof that (4.19) always has a solution proceeds along the same lines as in the proof given in Goovaerts et al. (2004). Theorems 4.5 and 4.6 derive necessary and sufficient conditions for two generalized Haezendonck–Goovaerts risk measures to be comparable.

Theorem 4.5. *Let φ be a continuous and strictly increasing function in $\mathbb{R}^+$ and h a valid distortion function. A necessary and sufficient condition such that $\rho_{\varphi,h}(X,t)$ is comparable with and larger than $\rho_{l,h}(X,t)$ (l means a linear Young function φ) is that φ is a convex function.*

Proof. For the ease of notation, we will use $\rho_{\varphi,h} = \rho_{\varphi,h}(X,t)$ and $\rho_{l,h} = \rho_{l,h}(X,t)$. If we assume that $\rho_{\varphi,h} > \rho_{l,h}$, then we get

$$1-\alpha = \varphi(1-\alpha) = \mathrm{E}\left[\varphi\left(\frac{\left(F_X^{-1}(U)-F_X^{-1}(\alpha)\right)_+}{\rho_{l,h}}\frac{\rho_{l,h}}{\rho_{\varphi,h}}\right)h'(1-U)\right].$$

Since $\frac{\rho_{l,h}}{\rho_{\varphi,h}} < 1$ we get

$$1-\alpha < \mathrm{E}\left[\varphi\left(\frac{\left(F_X^{-1}(U)-F_X^{-1}(\alpha)\right)_+}{\rho_{l,h}}\right)h'(1-U)\right].$$

On the other hand one has (using the definition of ρ_l) that

$$1-\alpha = \mathrm{E}\left[\varphi\left(\frac{\left(F_X^{-1}(U)-F_X^{-1}(\alpha)\right)_+}{\rho_{l,h}}\right)h'(1-U)\right] = \varphi\left(\mathrm{E}\left[\frac{\left(F_X^{-1}(U)-F_X^{-1}(\alpha)\right)_+}{\rho_{l,h}}h'(1-U)\right]\right).$$

Hence $\mathrm{E}[\varphi(Z)] > \varphi(\mathrm{E}[Z])$ and φ is convex. In case φ is convex one immediately sees that $\rho_{\varphi,h} > \rho_{l,h}$, as for every l one gets the inequality

$$\mathrm{E}\left[\varphi\left(\frac{\left(F_X^{-1}(U)-F_X^{-1}(\alpha)\right)_+}{\rho}\right)h'(1-U)\right] > \varphi\left(\mathrm{E}\left[\frac{\left(F_X^{-1}(U)-F_X^{-1}(\alpha)\right)_+}{\rho}h'(1-U)\right]\right). \quad \square$$

Remark 4.1. It can be shown that a necessary and sufficient condition such that $\rho_{\varphi,h}(X,t)$ is comparable with and smaller than $\rho_{l,h}(X,t)$ is that φ is a concave function.

Theorem 4.6. *Let φ_1 and φ_2 be two continuous and strictly increasing functions on $\mathbb{R}$ and h a valid distortion function. A necessary and sufficient condition that $\rho_{\varphi_1,h}(X,t)$ is comparable with $\rho_{\varphi_2,h}(X,t)$ is that*

$$f = \varphi_2\varphi_1^{-1}$$

should satisfy

$$f(\mathrm{E}[X]) \leq \mathrm{E}[f(X)],$$

or the reversed inequality for all $X \in \mathcal{B}$.

Proof. Again, we use the following notation: $\rho_{\varphi_1,h} = \rho_{\varphi_1,h}(X,t)$ and $\rho_{\varphi_2,h} = \rho_{\varphi_2,h}(X,t)$. Suppose $\rho_{\varphi_2,h} \geq \rho_{\varphi_1,h}$, then we get

$$1-\alpha = \varphi_2(1-\alpha) = \mathrm{E}\left[\varphi_2\left(\frac{\left(F_X^{-1}(U)-F_X^{-1}(\alpha)\right)_+}{\rho_{\varphi_1,h}}\frac{\rho_{\varphi_1,h}}{\rho_{\varphi_2,h}}\right)h'(1-U)\right].$$

But, since $\frac{\rho_{\varphi_1,h}}{\rho_{\varphi_2,h}} < 1$ we get

$$1-\alpha < \mathrm{E}\left[\varphi_2\left(\frac{\left(F_X^{-1}(U)-F_X^{-1}(\alpha)\right)_+}{\rho_{\varphi_1,h}}\right)h'(1-U)\right].$$

On the other hand one has (using the definition of $\rho_{\varphi_1,h}$) that

$$1-\alpha = \mathrm{E}\left[\varphi_1\left(\frac{\left(F_X^{-1}(U)-F_X^{-1}(\alpha)\right)_+}{\rho_{\varphi_1,h}}\right)h'(1-U)\right].$$

Consequently,

$$\mathrm{E}\left[\varphi_2\varphi_1^{-1}\varphi_1\left(\frac{\left(F_X^{-1}(U)-F_X^{-1}(\alpha)\right)_+}{\rho_{\varphi_1,h}}\right)h'(1-U)\right] \geq \varphi_2\varphi_1^{-1}\mathrm{E}\left[\varphi_1\left(\frac{\left(F_X^{-1}(U)-F_X^{-1}(\alpha)\right)_+}{\rho_{\varphi_1,h}}\right)h'(1-U)\right].$$

Hence, φ_2 is a convex function in φ_1. $\square$

The classical properties remain the same after the introduction of the distortion function h.

5. Subadditivity, comonotonicity and monotonicity

Because φ, used in the Haezendonck–Goovaerts risk measure, is non-decreasing, it follows immediately that $X <_{st} Y$ implies $\rho_\varphi(X,t) \geq \rho_\varphi(Y,t)$. This is a direct consequence of the choice of a mean value principle. As far as comparability is concerned, this property is a direct consequence of the convexity of the function φ. It is remarkable that the convexity of φ is also a consequence of the introduction of a comonotone additive risk measure with a concave distortion function. Hence comparability and concave distortion functions both imply the convexity of the function φ. We also have that the stop-loss order is preserved: $X <_{sl} Y$ implies $\rho_\varphi(X,t) \geq \rho_\varphi(Y,t)$.

In the framework of the Haezendonck–Goovaerts risk measure, the following axioms are "equivalent" in the sense that they imply convexity of the function φ:

1. $X <_{\mathrm{cx}} Y \Longrightarrow \rho_\varphi(X,t) \geq \rho_\varphi(Y,t)$ implies that φ is convex.
2. In case φ is determined such that for a two point risk the resulting risk measure is concave, φ is convex.
3. Comparability of a Haezendonck–Goovaerts risk measure and the conditional tail expectations implies that φ has to be convex.

Because φ is convex, we have that

$$\mathrm{E}\left[\varphi\left(\frac{X_1+X_2-t}{\rho-t}\right)\right] \leq \mathrm{E}\left[\varphi\left(\frac{X_1^c+X_2^c-t}{\rho-t}\right)\right],$$

for all convex functions φ.

Consequently for all values of t, one gets that (equated to give $1-\alpha$) that

$$\rho_\varphi(X_1+X_2) \leq \rho_\varphi\left(X_1^c+X_2^c\right).$$

Hence the Haezendonck–Goovaerts risk measure preserves stop-loss order. Because $\rho_\varphi(X_1) = \rho_\varphi\left(X_1^c\right)$ and $\rho_\varphi(X_2) = \rho_\varphi\left(X_2^c\right)$, one gets the following inequalities:

$$\begin{aligned}
&\mathrm{E}\left[\varphi\left(\frac{\left(X_1^c+X_2^c-t_1-t_2\right)_+}{\rho_\varphi\left(X_1^c\right)+\rho_\varphi\left(X_2^c\right)-t_1-t_2}\right)\right] \\
&\leq \mathrm{E}\left[\varphi\left(\frac{\left(X_1^c-F_{X_1}^{-1}(\beta)\right)_+ + \left(X_2^c-F_{X_2}^{-1}(\beta)\right)_+}{\rho_\varphi\left(X_1^c\right)-F_{X_1}^{-1}(\beta)+\rho_\varphi\left(X_2^c\right)-F_{X_2}^{-1}(\beta)}\right)\right] \\
&\leq \sum_{j=1}^{2}\frac{\rho_\varphi\left(X_j^c\right)-F_{X_j}^{-1}(\beta)}{\sum_{i=1}^{2}\rho_\varphi\left(X_i^c\right)-F_{X_i}^{-1}(\beta)}\mathrm{E}\left[\varphi\left(\frac{\left(X_j^c-F_{X_j}^{-1}(\beta)\right)_+}{\rho_\varphi\left(X_j^c\right)-F_{X_j}^{-1}(\beta)}\right)\right] \\
&\leq 1-\alpha.
\end{aligned}$$

Hence

$$\rho_\varphi\left(X_1^c + X_2^c\right) \le \rho_\varphi\left(X_1^c\right) + \rho_\varphi\left(X_2^c\right).$$

When one starts with a discrete random variable X, the following theorem extends the concept of subadditive measures that is deduced by Huber (1981) for continuous distribution functions. Indeed, for any discrete random variable, $X \overset{d}{=} F_X^{-1}(U)$, with $U \sim$ uniform $(0, 1)$, one can write a claim X as the sum of comonotonic Bernoulli random variables with q_i the probabilities of the jumps smaller than $1 - \alpha$. For a random variable without jumps, one immediately has that the probability of a jump is smaller than $1-\alpha$ (for nonzero α). The result for the continuous (no jumps) random variables immediately follows as a limiting case from the case where jumps are allowed, taking into account the particular value of α.

Theorem 5.1. *Let us consider a discrete cumulative distribution function after the inclusion of a shift t at the origin. In fact, one considers a discrete random variable $(X - t)_+$ with*

$$\Pr\left((X-t)_+ = a_i\right) = q_i, \quad i = 1, 2, \dots, n.$$

Suppose the function φ is determined by a concave distortion function g such that

$$\frac{1-\alpha}{q} = \varphi\left(\frac{1}{g(q)}\right),$$

for $q < 1 - \alpha$ and $\Pr\left((X-t)_+ = 0\right) = 1 - \sum_{i=1}^n q_i$. Then, the Haezendonck–Goovaerts risk measure with a convex function φ provides an upper bound for the distortion risk measure.

Proof. Define the Bernoulli random variable $X_{(a_i - a_{i-1})q_i}$ as follows:

$$\Pr\left(X_{(a_i-a_{i-1})q_i} = a_i - a_{i-1}\right) = q_i,$$

$$\Pr\left(X_{(a_i-a_{i-1})q_i} = 0\right) = 1 - q_i.$$

In this case,

$$F_X^{-1}(U) \overset{d}{=} \sum_{i=1}^n X_{(a_i-a_{i-1})q_i} \overset{d}{=} \sum_{i=1}^n F^{-1}_{X_{(a_i-a_{i-1})q_i}}(U).$$

The distortion risk measure with distortion function g then can be put in the form

$$\rho_l\left((X-b)_+\right) = \sum_{i=1}^n (a_i - a_{i-1})\, g(q_i).$$

Consequently

$$\mathrm{E}\left[\varphi\left(\frac{\sum_{i=1}^n F^{-1}_{X_{(a_i-a_{i-1})q_i}}(U)}{\sum_{i=1}^n (a_i-a_{i-1})\, g(q_i)}\right)\right] \ge \sum_{i=1}^n \frac{(a_i-a_{i-1})\, g(q_i)}{\sum_{j=1}^n \left(a_j - a_{j-1}\right) g\left(q_j\right)} \mathrm{E}\left[\varphi\left(\frac{F^{-1}_{X_{(a_i-a_{i-1})q_i}}(U)}{(a_i-a_{i-1})\, g(q_i)}\right)\right] = \sum_{i=1}^n \frac{(a_i-a_{i-1})\, g(q_i)}{\sum_{j=1}^n \left(a_j - a_{j-1}\right) g\left(q_j\right)} q_i \varphi\left(\frac{1}{g(q_i)}\right) = 1 - \alpha.$$

Hence

$$\mathrm{E}\left[\varphi\left(\frac{\sum_{i=1}^n F^{-1}_{X_{(a_i-a_{i-1})q_i}}(U)}{\rho_\varphi\left((X-t)_+\right)}\right)\right] = 1 - \alpha,$$

entails that $\rho_\varphi\left((X-t)_+\right)$ is larger than the distortion risk measure. □

6. Conclusion

In this contribution we introduce the Haezendonck–Goovaerts risk measure as an application of the mean value principle. In addition, the distortion function applied to a Bernoulli risk provides us with an interpretation of the function φ, appearing in Haezendonck–Goovaerts risk measure applied to the calculation of a solvency margin. It is shown that each of the supplementary axioms (1) subadditivity of the risk measures, (2) respecting stop loss order, (3) comparability of the solvency margin for a given risk X, separately require convexity of the function φ. The random variables considered might show jumps when the probability of a jump is smaller than $1 - \alpha$ in order to trace back the axiomatic approach of Huber.

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