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Fair valuation of insurance liabilities: merging actuarial judgement and market-consistency

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Abstract

In this paper, we investigate a single period framework for the fair valuation of the liabilities related to an insurance policy or portfolio. We define a fair valuation as a valuation which is both market-consistent (mark-to-market for hedgeable claims) and actuarial (mark-to-model for unhedgeable claims). We introduce the class of hedge-based valuations, where in a first step of the valuation process, a 'best hedge' for the liability is set up, based on the traded assets in the market, while in a second step, the remaining part of the claim is valuated via an actuarial valuation. We also introduce the class of two-step valuations, the elements of which are closely related to the two-step valuations which were introduced in Pelsser and Stadje (2014). We show that the 3 classes of fair, hedge-based and two-step valuations are identical.

Keywords: Fair valuation of insurance liabilities, market-consistent valuation, actuarial valuation, Solvency II, mean-variance hedging.

1 Introduction

Modern solvency regulations for the insurance industry, such as the Swiss Solvency Test and Solvency II, require insurance undertakings to apply a *fair valuation* of their assets and liabilities. The fair value of an asset or a liability is generally understood as 'the amount for which it could be transferred (exchanged) between knowledgeable willing parties in an arm's length transaction'. A fair valuation method should combine techniques from financial mathematics and actuarial science, in order to take into account and be

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consistent with information provided by the financial markets on the one hand and actuarial judgement based on generally available data concerning the underlying risks on the other hand.[1]

This paper is about the generic meaning of fair valuation of random payments related to liabilities in an insurance context and not about a particular technical meaning that is given to it by a particular regulation or legislation. Furthermore, we consider fair valuation in a general context without specifying the purpose it is used for. The results we present and discuss may be used not only in a reserving context (determining technical provisions) but also in a pricing context (setting premiums).

Consider a set of future random payoffs, further called (*contingent*) *claims*. Some of these claims are traded, i.e. they can be bought and sold in a financial market. We assume that the market of traded claims is incomplete. This means that apart from hedgeable claims, of which the fair value is equal to the price of its underlying hedge, there are also claims that cannot be hedged. Insurance claims typically belong to this class of unreplicable claims.

Several ways of valuating unhedgeable (unreplicable) claims have been considered in the literature. Under a 'utility indifference' approach, the value of the insurance liability is set equal to the amount such that the insurer is indifferent between selling the insurance or not in terms of expected utility. The idea for the utility indifference approach is often attributed to Hodges and Neuberger (1989). A market-consistent insurance premium based on expected utility indifference arguments is developed in Malamud et al. (2008). A similar algorithm was proposed by Musiela and Zariphopoulou (2004) for determining indifference prices in a multiperiod binomial model. For an overview of the theory, we refer to Henderson and Hobson (2004) and Carmona (2009).

Another approach for valuating unreplicable claims starts from the observation that in an incomplete market setting no-arbitrage arguments only partially specify the pricing measure (which allows to express prices of contingent claims as discounted expectations under that measure). Therefore, one extends this partially specified measure to a 'complete' pricing measure that is used to determine the value of all contingent claims, also the ones that are not traded. The 'complete' pricing measure is chosen such that it is, in one way or another, the 'most appropriate' one. A popular choice is the minimal entropy martingale measure, see e.g. Frittelli (1995) and Frittelli (2000) in a pure financial framework or Dhaene et al. (2015) in a combined financial-actuarial framework. Another possible choice is the risk-neutral Esscher measure, see Gerber and Shiu (1994). Under such a 'completing approach', the value of an unhedgeable claim is in fact a reasoned estimate of what its market value would have been had it been readily traded.

A market-consistent valuation is usually defined in terms of an extension of the notion of cash invariance to all hedgeable claims, see e.g. Malamud et al. (2008), Artzner and

[1] Solvency II (Directive 2009/138/EC, Article 77) states that 'if the cash flows of the liability (or part of the cash flows) can be replicated reliably using financial instruments for which a reliable market value is observable, then the value of the (part of the) cash flows is determined on the basis of the market value of these financial instruments. Otherwise, the value is equal to the sum of the best estimate and a risk margin'.

Eisele (2010) or Pelsser and Stadje (2014). We will define a *fair valuation* as a valuation which is both market-consistent and actuarial. This valuation is *market-consistent* in the sense that hedgeable claims are valuated at the price of their hedge. Moreover, the valuation is *actuarial* in the sense that claims with payoffs that are independent of the evolution of asset prices are valuated taking into account actuarial judgement. We show that actuarial, market-consistent and fair valuations can be characterized in terms of hedging strategies.

We introduce and investigate 'hedge-based valuations'. Under this approach, one unbundles the unhedgeable insurance claim in a hedgeable part and a remaining part. The fair value of the claim is then set equal to the sum of the respective values of the hedgeable and the unhedgeable parts, where the hedgeable part is valued by the financial price of its underlying hedge, while the remaining part is valuated via an actuarial approach. In particular, we consider 'convex hedge-based valuations'. As a special case, we investigate 'mean-variance hedge-based valuations'. Further, we also consider an adapted version of the two-step valuation approach, as introduced in Pelsser and Stadje (2014). We will show that any fair valuation can be expressed as a hedge-based valuation as well as a two-step valuation.

Over the last two decades, several researchers have worked on the unification of the two fields of financial and actuarial valuation. This research area is still in full development and the literature on market-consistent valuation is growing rapidly. An early overview of the different aspects of the interplay between the two fields is given in Embrechts (2000). In Delong (2011), the author deals with practical and theoretical aspects of market-consistent valuation by discussing advanced mathematical methods to price insurance liabilities. Other interesting references include Møller (2001), Barrieu and El Karoui (2005), Barrieu and El Karoui (2009), Knispel et al. (2011), Kupper et al. (2008), Malamud et al. (2008), Møller (2001), Pelsser and Ghalehjooghi (2016*b*), Pelsser and Ghalehjooghi (2016*a*), Pelsser and Ghalehjooghi (2016*c*), Pelsser and Stadje (2014), Tsanakas et al. (2013), Wüthrich et al. (2010) and the references therein.

The remainder of the paper is structured as follows. In Section 2, we describe the financial-actuarial world and its market of traded assets. In Section 3, fair valuations and the related notion of fair hedgers are introduced. Hedge-based valuations are considered in Section 4. A generalization of the two-step valuations introduced by Pelsser and Stadje (2014) is considered in Section 5. Section 6 concludes the paper.

2 The financial-actuarial world and its market of traded assets

In this paper, we investigate the fair valuation of traded and non-traded payoffs in a single period financial-actuarial world. Let time 0 be 'now' and consider a set of random payoffs, which are due at time 1. These payoffs are random variables (r.v.'s) defined on a given probability space $(\Omega, \mathcal{G}, \mathbb{P})$, which is a mathematical abstraction of the combined financial-actuarial world. We call the random payoffs *contingent claims* or also *claims*.

Throughout the paper, we assume that the second moments of all claims and the first moments of all products of claims that we will encounter exist under $\mathbb{P}$.

Any element of $\omega \in \Omega$ represents a possibly state of the financial-actuarial world at time 1. For instance, each ω could represent a set of possible outcomes of the time-1 prices of the stocks composing the Dow Jones Index and the number of survivors at time 1 from a given closed population observed at time 0. The σ-algebra $\mathcal{G}$ is the set of all events that may or may not occur in this single period world. Probabilities for these events follow from the real-world probability measure $\mathbb{P}$. We denote the set of all (contingent) claims defined on $(\Omega, \mathcal{G})$, that is the set of all $\mathcal{G}-$measurable r.v.'s, by $\mathcal{C}$.

The financial-actuarial world $(\Omega, \mathcal{G}, \mathbb{P})$ is home to a market of $n+1$ traded assets. These assets can be bought or sold in any quantities in a deep, liquid, transparent and frictionless market (no transaction costs and other market frictions). Asset 0 is the risk-free zero coupon bond. This asset has current price $y^{(0)} = 1$, while its payoff at time 1 is given by $Y^{(0)} = e^r$, where $r \geq 0$ is the (continuously compounded) deterministic interest rate r. Furthermore, there are n risky assets, denoted by $1, \ldots, n$, traded in the market. The price (or the payoff) at time 1 of each asset is a claim defined on $(\Omega, \mathcal{G})$. The current price of asset $m \in \{1, 2, \ldots, n\}$, is denoted by $y^{(m)} > 0$, whereas its non-deterministic payoff at time 1 is $Y^{(m)} \geq 0$. Each asset m is characterized by the process $\left(y^{(m)},\ Y^{(m)}\right)$ defined on $(\Omega, \mathcal{G})$. We introduce the notations $\mathbf{y}$ and $\mathbf{Y}$ for the vectors of the time-0 and time-1 asset prices, respectively:

$$\boldsymbol{y} = \left(y^{(0)}, y^{(1)}, \ldots, y^{(n)}\right)$$

and

$$\boldsymbol{Y} = \left(Y^{(0)}, Y^{(1)}, \ldots, Y^{(n)}\right).$$

A *trading strategy* $\boldsymbol{\theta} = \left(\theta^{(0)}, \theta^{(1)}, \ldots, \theta^{(n)}\right)$ is an $(n+1)-$dimensional real-valued vector, where the quantity $\theta^{(m)}$ stands for the number of units invested in asset m at time 0. The time-0 and time-1 values of the trading strategy $\boldsymbol{\theta}$ are given by the scalar products

$$\boldsymbol{\theta} \cdot \boldsymbol{y} = \sum_{m=0}^{n} \theta^{(m)}\ y^{(m)}$$

and

$$\boldsymbol{\theta} \cdot \boldsymbol{Y} = \sum_{m=0}^{n} \theta^{(m)}\ Y^{(m)},$$

respectively. The set of all trading strategies $\boldsymbol{\theta} = \left(\theta^{(0)}, \theta^{(1)}, \ldots, \theta^{(n)}\right)$ is denoted by Θ.

Throughout the paper, we assume that the $n+1$ assets are non-redundant, which means that there exists no investment strategy $\boldsymbol{\theta}$ which is different from $\mathbf{0} = (0, 0, \ldots, 0)$ such that $\boldsymbol{\theta} \cdot \boldsymbol{Y} = 0$. Hence,

$$\boldsymbol{\theta} \cdot \boldsymbol{Y} = 0 \Rightarrow \boldsymbol{\theta} = \mathbf{0}. \tag{1}$$

By convention, (in-)equalities between r.v.'s, such as $\boldsymbol{\theta} \cdot \boldsymbol{Y} = 0$, have to be understood in the $\mathbb{P}$-almost sure sense, unless explicitly stated otherwise. The non-redundancy assumption implies that the time-1 value $\boldsymbol{\theta} \cdot \boldsymbol{Y}$ of a trading strategy uniquely determines the trading strategy $\boldsymbol{\theta}$.

A probability measure $\mathbb{Q}$ defined on the measurable space $(\Omega, \mathcal{G})$ is said to be an *equivalent martingale measure* (or a risk-neutral measure), further abbreviated as EMM, for the market defined above, if it fulfills the following conditions:

(1) $\mathbb{Q}$ and $\mathbb{P}$ are equivalent probability measures:

$$\mathbb{P}[A] = 0 \text{ if and only if } \mathbb{Q}[A] = 0, \quad \text{for all } A \in \mathcal{G}.$$

(2) The current price of any traded asset in the market is given by the expected value of the discounted payoff of this asset at time 1, where discounting is performed at the risk-free interest rate r and expectations are taken with respect to $\mathbb{Q}$:

$$y^{(m)} = e^{-r} \, \mathbb{E}^{\mathbb{Q}}\left[Y^{(m)}\right], \quad \text{for } m = 0, 1, ..., n.$$

Hereafter, we always assume that the market is arbitrage-free in the sense that there is no investment strategy $\boldsymbol{\theta} \in \Theta$ such that

$$\boldsymbol{\theta} \cdot \boldsymbol{y} = 0, \quad \mathbb{P}[\boldsymbol{\theta} \cdot \mathbf{Y} \geq 0] = 1 \quad \text{and} \quad \mathbb{P}[\boldsymbol{\theta} \cdot \mathbf{Y} > 0] > 0.$$

It is well-known that in our setting, the no-arbitrage condition is equivalent to the existence of a (not necessarily unique) equivalent martingale measure, whereas completeness of the arbitrage-free market is equivalent to the existence of a unique equivalent martingale measure, see e.g. Dalang et al. (1990).

Definition 1 (Hedgeable claim) *A hedgeable claim S^h is an element of $\mathcal{C}$ that can be replicated by a trading strategy $\boldsymbol{\nu} = \left(\nu^{(0)}, \nu^{(1)}, \ldots, \nu^{(n)}\right) \in \Theta$:*

$$S^h = \boldsymbol{\nu} \cdot \mathbf{Y} = \sum_{m=0}^{n} \nu^{(m)} \, Y^{(m)}. \tag{2}$$

We introduce the notation $\mathcal{C}^h$ for the set of all hedgeable claims. The time-0 price of $S^h = \boldsymbol{\nu} \cdot \mathbf{Y}$ is given by

$$\boldsymbol{\nu} \cdot \boldsymbol{y} = \sum_{m=0}^{n} \nu^{(m)} \, y^{(m)} = e^{-r} \, \mathbb{E}^{\mathbb{Q}}\left[S^h\right], \tag{3}$$

where $\mathbb{Q}$ is a generic member of the class of EMM's. .

Definition 2 (Orthogonal claim) *An orthogonal claim $S^\perp$ is an element of $\mathcal{C}$ which is $\mathbb{P}$-independent of all traded claims $Y^{(m)}$, $m = 1, \ldots, n$:*

$$S^\perp \perp \left(Y^{(1)}, Y^{(2)}, \ldots Y^{(n)}\right). \tag{4}$$

Hereafter, we will denote the set of all orthogonal claims by $\mathcal{C}^\perp$. The risk-free claims $a \in \mathbb{R}$ are the only claims which are both hedgeable and orthogonal. Obviously, the hedge related to the claim a due at time 1 is an investment of amount $e^{-r}a$ in zero coupon bonds.

Example 1 [**Cost-of-capital principle for orthogonal liabilities**]
Consider the liability $S^{\perp}$ related to a portfolio of one-year insurances:

$$S^{\perp} = \sum_{i=1}^{N} X_i,$$

where $X_1, X_2, \ldots X_N$ are the losses of the different policies, which are assumed to be $\mathbb{P}$-independent of $\left(Y^{(1)}, Y^{(2)}, \ldots Y^{(n)}\right)$. The position of the insurer in the orthogonal liability $S^{\perp}$ cannot be hedged in the financial market. Suppose that the regulator requires the holder of this position to set up a provision $e^{-r}\mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right]$ and a capital buffer $\left(VaR_p\left[S^{\perp}\right] - \mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right]\right)$, for some probability level p. The fair value $\pi\left[S^{\perp}\right]$ of $S^{\perp}$ is then determined in the following way:

$$\pi\left[S^{\perp}\right] = \text{BE}\ \left[S^{\perp}\right] +\ \text{RM}\ \left[S^{\perp}\right], \tag{5}$$

where BE $\left[S^{\perp}\right]$ is the best estimate of $S^{\perp}$, which is given by

$$\text{BE}\ \left[S^{\perp}\right] = e^{-r}\mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right],$$

while RM $\left[S^{\perp}\right]$ is the risk margin, i.e. the amount to be added to the best estimate, so that another party would be willing to take over $S^{\perp}$. Under the cost-of-capital approach, the risk margin is determined by

$$\text{RM}\ \left[S^{\perp}\right] = e^{-r}\ i\ \left(\text{VaR}_p\left[S^{\perp}\right] - \mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right]\right),$$

for some perunage i (called the cost-of-capital rate). In this case, the risk margin reflects the cost related to holding the capital $\left(\text{VaR}_p\left[S^{\perp}\right] - \mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right]\right)$ to buffer the risk of $S^{\perp}$ being larger than its best estimate. As $\pi\left[S^{\perp}\right]$ is based on actuarial considerations (concerning the choice of the $\mathbb{P}$-measure and the risk margin), it is not uniquely determined. Actuaries may have different opinions about the value of an orthogonal claim. Let us now additionally assume that under $\mathbb{P}$, the claims X_i are i.i.d. with expectation and variance given by μ and $\sigma^2 > 0$, respectively.
Let the portfolio be sufficiently large such that we can assume that

$$\mathbb{P}\left[\frac{S^{\perp} - \mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right]}{\sigma^{\mathbb{P}}\left[S^{\perp}\right]} \le s\right] = \Phi\left[s\right], \qquad \text{for all } s,$$

where Φ is the standard normal distribution. In this case, we find that $\pi\left[S^{\perp}\right]$ is given by

$$\pi\left[S^{\perp}\right] = e^{-r}\left(N\mu + i\sqrt{N}\sigma\Phi^{-1}\left[p\right]\right), \tag{6}$$

which shows that the fair value per policy

$$\frac{\pi\left[S^{\perp}\right]}{N} = e^{-r}\left(\mu + i\frac{\sigma}{\sqrt{N}}\Phi^{-1}\left[p\right]\right) \tag{7}$$

decreases if the portfolio size N increases. ◀

Many claims that insurance companies face are not perfectly hedgeable, but nevertheless not $\mathbb{P}$-independent of the payoffs of the traded assets. Such claims are neither hedgeable nor orthogonal. Instead, they belong to the class of unhedgeable and non-orthogonal claims. Hereafter, we will call the members of this class hybrid claims.

Definition 3 (Hybrid claim) *A claim S is called a hybrid claim in case it is neither perfectly hedgeable nor orthogonal:*

$$S \in \mathcal{C} \setminus \left(\mathcal{C}^h \cup \mathcal{C}^\perp\right).$$

Unit-linked insurance products often have by construction a financial (hedgeable) and an actuarial (unhedgeable) part in their payoff. This means that the valuation of unit-linked insurance claims gives rise to the valuation of hybrid claims. Furthermore, the development of markets in insurance-linked securities (such as catastrophic bonds, weather derivatives, longevity bonds) creates the possibility that liabilities of insurance portfolios that are exposed to specific actuarial risks (such as those arising from natural catastrophes) become at least partially hedgeable. Hence, insurance securitization may also lead to hybrid claims in insurance portfolios.

Insurance valuation regulations are in general clear about the fair valuation of hedgeable and orthogonal claims. The former type of claims are valuated at the cost of the replicating portfolio, while the latter are valuated as the sum of their expected present value and a risk margin. However, it is usually unclear how to perform the fair valuation of hybrid claims. This paper contributes to the development of solutions for that important issue.

3 Fair valuations and fair hedgers

In this section, we define different classes of valuations, which attach a value to any claim $S \in \mathcal{C}$. We also introduce different classes of hedgers, which attach a trading strategy to any such claim. We show that there is a one-to-one relation between each class of hedgers and its corresponding class of valuations.

3.1 Fair valuations

In this subsection, we define the notion of valuation. Furthermore, we also introduce the notions of market-consistent, actuarial and fair valuations, respectively.

Definition 4 (Valuation) *A valuation is a mapping $\rho : \mathcal{C} \to \mathbb{R}$, attaching a real number to any claim $S \in \mathcal{C}$:*

$$S \to \rho[S],$$

such that ρ is normalized:

$$\rho[0] = 0, \tag{8}$$

and ρ is translation invariant:

$$\rho[S+a] = \rho[S] + e^{-r}a, \qquad \text{for any } S \in \mathcal{C} \text{ and } a \in \mathbb{R}. \tag{9}$$

A valuation ρ attaches a real number to any claim, which we interprete as a 'value' of that claim. For any valuation ρ, we immediately find that

$$\rho[a] = e^{-r}a, \qquad \text{for any } a \in \mathbb{R}. \tag{10}$$

Other properties that a valuation may satisfy or not are $\mathbb{P}$-law invariance, positive homogeneity and subadditivity. A valuation ρ is said to be $\mathbb{P}$-*law invariant* if

$$\rho[S_1] = \rho[S_2] \text{ for any } S_1, S_2 \in \mathcal{C} \text{ with } S_1 \overset{\mathbb{P}}{=} S_2.$$

It is said to be *positive homogenous* if

$$\rho[aS] = a\ \rho[S], \qquad \text{for any scalar } a > 0 \text{ and any } S \in \mathcal{C},$$

while it is said to be *subadditive* if

$$\rho[S_1 + S_2] \leq \rho[S_1] + \rho[S_2], \qquad \text{for any } S_1, S_2 \in \mathcal{C}.$$

An important subclass of the class of valuations is the class of market-consistent valuations, which are defined hereafter.

Definition 5 (Market-consistent valuation) *A market-consistent valuation (MC valuation) is a valuation $\rho : \mathcal{C} \to \mathbb{R}$ such that any hedgeable part of any claim is mark-to-market:*

$$\rho[S + \boldsymbol{\nu} \cdot \mathbf{Y}] = \rho[S] + \boldsymbol{\nu} \cdot \boldsymbol{y}, \qquad \text{for any } S \in \mathcal{C} \text{ and any } \boldsymbol{\nu} \cdot \mathbf{Y} \in \mathcal{C}^h. \tag{11}$$

In the existing literature on MC valuation, market-consistency is usually defined via condition (11), see e.g. Kupper et al. (2008), Malamud et al. (2008) or Artzner and Eisele (2010) and Pelsser and Stadje (2014). The mark-to-market condition (11) can be interpreted as an extension of the notion of translation (or cash) invariance (9) from scalars to hedgeable claims. The mark-to-market condition can also be stated in the following way:

$$\rho[S] = \rho[S - \boldsymbol{\nu} \cdot \mathbf{Y}] + \boldsymbol{\nu} \cdot \boldsymbol{y}, \qquad \text{for any } S \in \mathcal{C} \text{ and } \boldsymbol{\nu} \cdot \mathbf{Y} \in \mathcal{C}^h. \tag{12}$$

In order to interpret (12), consider a person facing a loss S. This person could decide to transfer the whole loss S to the insurer. Alternatively, he could split his claim S into a hedgeable claim $\boldsymbol{\nu} \cdot \mathbf{Y}$, which he hedges in the financial market, while he brings the remaining part $S - S^h$ to the insurer. The condition (12) states that the claim S is equally valuated in both cases. In other words, the insurer valuates in agreement with the financial market, in the sense that he does not charge a risk margin for any hedgeable part of a claim. From (12), we also find that for any hedgeable claim $S^h = \boldsymbol{\nu} \cdot \mathbf{Y}$, we have that

$$\rho\left[\boldsymbol{\nu} \cdot \mathbf{Y}\right] = \boldsymbol{\nu} \cdot \boldsymbol{y}. \tag{13}$$

which means that the MC value of a hedgeable claim is equal to the price of its underlying hedge.

Next we define actuarial valuations. Notice that the notation π will be used exclusively to denote an actuarial valuation, while ρ is used to represent any type of valuation.

Definition 6 (Actuarial valuation) *An actuarial valuation is a valuation $\pi : \mathcal{C} \rightarrow \mathbb{R}$ such that any orthogonal claim is mark-to-model:*

$$\pi\left[S^{\perp}\right] \text{ does not depend on } \left(y^{(1)}, y^{(2)}, \ldots, y^{(n)}\right), \qquad \text{for any } S^{\perp} \in \mathcal{C}^{\perp}. \tag{14}$$

The mark-to-model condition (14) states that in the set of orthogonal claims, π does not depend on the current prices of traded assets. The actuarial valuation π is chosen by the actuary, the regulator or any other valuator of the claims. It introduces actuarial considerations in the valuation of claims. The condition (14) corresponds to the traditional view on valuation in an insurance context that ignores the existence of a financial market, except for the risk-free bank account. In such a model, any claim S is orthogonal, and the value of any S is given by $\pi\left[S\right]$.

Our definition of an actuarial valuation is kept very general. Notice however that all results that we will derive hereafter remain valid if we adapt the definition of an actuarial valuation by requiring another property for the valuation of orthogonal claims. One might require e.g. that the actuarial valuation is $\mathbb{P}$-law invariant for orthogonal claims. Or one might consider a set of probability measures $(\mathbb{P}_1, \mathbb{P}_2, \ldots, \mathbb{P}_n)$ on the measurable space $(\Omega, \mathcal{G})$, and require that for any orthogonal claim $S^{\perp} \in \mathcal{C}^{\perp}$, $\pi\left[S^{\perp}\right]$ only depends on the n cdf's $F_{S^{\perp}}^{\mathbb{P}_1}, F_{S^{\perp}}^{\mathbb{P}_2}, , \ldots, F_{S^{\perp}}^{\mathbb{P}_n}$ of $S^{\perp}$ under these different measures. An example is the 'worst-case' valuation

$$\pi\left[S^{\perp}\right] = \max\left(\pi^{\mathbb{P}_1}\left[S^{\perp}\right], \pi^{\mathbb{P}_2}\left[S^{\perp}\right], \ldots, \pi^{\mathbb{P}_n}\left[S^{\perp}\right]\right), \qquad \text{for any } S^{\perp} \in \mathcal{C}^{\perp},$$

where for each i, $\pi^{\mathbb{P}_i}$ is the standard deviation premium principle under the measure $\mathbb{P}_i$.

Definition 7 (Fair valuation) *A fair valuation is a valuation that is both market-consistent and actuarial.*

Current insurance solvency regulations impose a mark-to-market as well as a mark-to-model requirement for the valuation of assets and liabilities.[2]. However, in the existing

[2]In the 'Solvency II Glossary' of the 'Comité Européen des Assurances' and the 'Groupe Consultatif Actuariel Européen' of 2007, market-consistent valuation is defined as 'the practice of valuing assets and liabilities on market values where observable with a given quality (mark-to-market), where not, on market-consistent valuation techniques (mark-to-model). In that same publication, mark-to-market valuation and mark-to-model valuation are defined as the practice of valuing insurance rights and obligations, or more broadly security and financial instruments, 'using current market prices' and 'based on modelling', respectively.

scientific literature on valuating claims in a combined financial-actuarial setting, the focus is on the mark-to-market condition as defined according to (11), while the mark-to-model condition, which states that non-financial components of a claim should be valued taking into account actuarial judgement and experience, is ignored. Therefore, we introduce fair valuations, which are a subset of the class of market-consistent valuations. These fair valuations are closer to the meaning of market-consistent or fair valuation in current insurance solvency regulations, as they satisfy a mark-to-market as well as a mark-to-model condition.

At first sight, one could wonder whether it would be more appropriate to define a fair valuation as a valuation that obeys the mark-to-market condition (11) for any hedgeable part of a claim, as well as the following mark-to-model conditon for any orthogonal part of a claim:

$$\rho\left[S + S^{\perp}\right] = \rho\left[S\right] + \pi\left[S^{\perp}\right], \quad \text{for any } S \in \mathcal{C} \text{ and any } S^{\perp} \in \mathcal{C}^{\perp},$$

where π is an actuarial valuation as defined above. One can easily prove that this condition would imply that $\rho\left[S^{\perp}\right] = \pi\left[S^{\perp}\right]$ and hence,

$$\pi\left[S_1^{\perp} + S_2^{\perp}\right] = \pi\left[S_1^{\perp}\right] + \pi\left[S_2^{\perp}\right], \quad \text{for any } S_1^{\perp},\, S_2^{\perp} \in \mathcal{C}^{\perp},$$

which would ignore the diversification benefit which is essential for valuating non-replicable insurance liabilities, see e.g. (6) in Example 1.

The valuation $\rho : \mathcal{C} \to \mathbb{R}$ defined by

$$\rho\left[S\right] = \mathbb{E}^{\mathbb{Q}}\left[S\right] \tag{15}$$

for a given EMM $\mathbb{Q}$ is an example of a valuation which is market-consistent but in general not actuarial and hence, not fair. Using the risk neutral valuation (15) for hybrid and orthogonal claims again ignores the diversification effect in insurance portfolios. Consider e.g. the orthogonal claim $S^{\perp} = \sum_{i=1}^{N} X_i$, where the claims X_i are i.i.d. and suppose that this claim is valuated by (15). The value per policy is then given by

$$\frac{\rho\left[S^{\perp}\right]}{N} = e^{-r}\, \mathbb{E}^{\mathbb{Q}}\left[X_1\right],$$

which is independent of the size N of the portfolio and hence, does not take into account the diversification effect.

From the requirements (11) and (14), we find that the fair valuation for $S^{\perp} + \boldsymbol{\nu} \cdot \mathbf{Y}$ with $S^{\perp} \in \mathcal{C}^{\perp}$ and $\boldsymbol{\nu} \cdot \mathbf{Y} \in \mathcal{C}^{h}$ is given by

$$\rho\left[S^{\perp} + \boldsymbol{\nu} \cdot \mathbf{Y}\right] = \pi\left[S^{\perp}\right] + \boldsymbol{\nu} \cdot \mathbf{y}, \tag{16}$$

Hence, the MC value of the claim $S^{\perp} + S^{h}$ is given by the sum of the actuarial value of $S^{\perp}$ and the financial market price of $\boldsymbol{\nu} \cdot \mathbf{Y}$. In other words, the orthogonal part of the claim is mark-to-model, whereas the hedgeable part is mark-to-market.

Most hybrid claims observed in an insurance context are of a more complex structure than the additive structure considered in (16). Often, one will encounter a multiplicative structure, where the claim S to be valuated can be expressed in the form

$$S = S^h \times S^\perp, \quad \text{with } S^h \in \mathcal{C}^h \text{ and } S^\perp \in \mathcal{C}^\perp. \tag{17}$$

Solvency regulations are in general rather vague on how to evaluate such hybrid claims. It is obvious that this claim is only partially hedgeable, and that S^h is hedgeable whereas $S^\perp$ is not. But it is not clear how to combine market prices of hedgeable claims with actuarial considerations to determine a fair value for the claim, since regulatory frameworks usually do not prescribe how to determine the hedgeable part of a non-hedgeable claim.

Example 2 [**Unit-linked insurance**] Consider an insurance portfolio consisting of N insureds with remaining lifetimes $T_1, T_2, \ldots, T_N$ which are assumed to be i.i.d. and independent of $Y^{(1)}$. Suppose also that the put option with payoff $\left(K - Y^{(1)}\right)_+$, for some prefixed strike price $K \geq 0$, is traded. Further, suppose that each insured i has underwritten a one-year *unit-linked contract with guarantee* against the risk that asset 1 falls short of K. The payoff of individual contract i at time 1 is given by

$$\max\left(Y^{(1)}, K\right) \times 1_{\{T_i > 1\}}.$$

This unit-linked contract has an unbounded upside potential and offers downward protection. The payoff can also be expressed as follows:

$$\left[Y^{(1)} + \left(K - Y^{(1)}\right)_+\right] \times 1_{\{T_i > 1\}}.$$

In case $K = 0$, the contract is called a *pure unit-linked contract.* The portfolio liability is given by (17),

$$S^h \times S^\perp = \left[Y^{(1)} + \left(K - Y^{(1)}\right)_+\right] \times \sum_{i=1}^{N} 1_{\{T_i > 1\}}, \tag{18}$$

with S^h and $S^\perp$ given by

$$S^h = \left[Y^{(1)} + \left(K - Y^{(1)}\right)_+\right]$$

and

$$S^\perp = \sum_{i=1}^{N} 1_{\{T_i > 1\}},$$

respectively. ◀

More complicated hybrid claims arise when the claim S is given by

$$S = S^h \times S', \quad \text{with } S^h \in \mathcal{C}^h \text{ and } S' \in \mathcal{C}, \tag{19}$$

where S^h and S are not assumed to be independent. As an example, consider the claim S defined in the previous example, where we do not assume independence between the remaining lifetimes at one hand and the stock value $Y^{(1)}$ at the other hand.

A major simplification for valuating the claim S defined in (17), originating from Brennan and Schwartz (1976), see also Brennan and Schwartz (1979a,b), arises if we assume that the claim $S^{\perp}$ is completely diversified, in the sense that

$$S^{\perp} = \mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right].$$

This assumption can be justified for very large portfolios of independent claims by the law of large numbers. Under this assumption of complete diversification, we find that $S^h \times \mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right]$ is a hedgeable claim, only containing financial uncertainty and hence, taking into account (13), we find that

$$\rho\left[S^h \times S^{\perp}\right] = \boldsymbol{\nu} \cdot \boldsymbol{y} \times \mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right]. \tag{20}$$

Taking into account that

$$\boldsymbol{\nu} \cdot \boldsymbol{y} = e^{-r}\ \mathbb{E}^{\mathbb{Q}}\left[S^h\right]$$

for any EMM $\mathbb{Q}$, we can transform the previous expression in the well-known Brennan & Schwartz - formula:

$$\rho\left[S^h \times S\right] = e^{-r}\ \mathbb{E}^{\mathbb{Q}}\left[S^h\right] \times \mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right]. \tag{21}$$

This 'actuarial diversification-based' approach does not answer the question of how to quantify hybrid claims of the form $S^h \times S^{\perp}$ in case the law of large numbers is not applicable for the insurance claim $S^{\perp}$. In this case, one is not able to 'average out' the insurance risk. Instead, one has to consider $S^h \times S^{\perp}$ as a claim in an incomplete market and come up with a valuation approach that reflects both financial and actuarial risk. Such valuation approaches will be considered in the following sections.

3.2 Fair hedgers

After having defined several classes of valuations, in this subsection we introduce the corresponding classes of hedgers. In particular, we will define market-consistent, actuarial and fair hedgers. We investigate the relation between each type of valuation and its corresponding hedger.

Definition 8 (Hedger) *A hedger is a function $\boldsymbol{\theta} : \mathcal{C} \to \Theta$ which maps any claim S into a trading strategy $\boldsymbol{\theta}_S = \left(\theta_S^{(0)}, \theta_S^{(1)}, \ldots, \theta_S^{(n)}\right)$.*

The mapping $\boldsymbol{\theta} : \mathcal{C} \to \Theta$ is called a hedger, whereas for any claim S, the trading strategy $\boldsymbol{\theta}_S$ is called a hedge for S. This hedge may be a partial or a perfect hedge. The value of the hedge $\boldsymbol{\theta}_S$ of S at time 1 is given by

$$\boldsymbol{\theta}_S \cdot \mathbf{Y} = \sum_{m=0}^{M} \theta_S^{(m)}\ Y^{(m)}, \tag{22}$$

whereas its time-0 value equals

$$\boldsymbol{\theta}_S \cdot \boldsymbol{y} = \sum_{m=0}^{M} \theta_S^{(m)}\ y^{(m)} = e^{-r}\ \mathbb{E}^{\mathbb{Q}}\left[\boldsymbol{\theta}_S \cdot \mathbf{Y}\right], \tag{23}$$

where $\mathbb{Q}$ can be any element of the class of EMM's.

Definition 9 *A hedger $\theta : \mathcal{C} \to \Theta$ is said to be*
• *positive homogeneous if*

$$\boldsymbol{\theta}_{a\ S} = a\ \boldsymbol{\theta}_S, \qquad \textit{for any scalar } a > 0 \textit{ and any } S \in \mathcal{C},$$

• *additive if*

$$\boldsymbol{\theta}_{S_1+S_2} = \boldsymbol{\theta}_{S_1} + \boldsymbol{\theta}_{S_2}, \qquad \textit{for any } S_1, S_2 \in \mathcal{C}.$$

Hereafter, we introduce the subclasses of market-consistent, actuarial and fair hedgers.

Definition 10 (Market-consistent, actuarial and fair hedger)
(1) A hedger is market-consistent in case any hedgeable part $\boldsymbol{\nu} \cdot \mathbf{Y}$ of any claim is hedged by ν:

$$\boldsymbol{\theta}_{S+\boldsymbol{\nu}\cdot\mathbf{Y}} = \boldsymbol{\theta}_S + \boldsymbol{\nu}, \qquad \textit{for any } S \in \mathcal{C} \textit{ and any } \boldsymbol{\nu} \cdot \mathbf{Y} \in \mathcal{C}^h. \tag{24}$$

(2) A hedger is actuarial in case any orthogonal claim is hedged risk-free via an actuarial valuation π:

$$\boldsymbol{\theta}_{S^\perp} = \left(\pi\left[S^\perp\right], 0, \ldots, 0\right), \qquad \textit{for any } S^\perp \in \mathcal{C}^\perp. \tag{25}$$

(3) A hedger is fair in case it is market-consistent and actuarial.

For any actuarial or fair hedger $\boldsymbol{\theta}$ with actuarial valuation π used to hedge claims in $\mathcal{C}^\perp$, we call π the underlying actuarial valuation of $\boldsymbol{\theta}$. The condition (24) in the definition of a market-consistent hedger can also be expressed as follows: for any hedgeable claim $\boldsymbol{\nu} \cdot \boldsymbol{Y}$ and any claim S, one has that

$$\boldsymbol{\theta}_S = \boldsymbol{\nu} + \boldsymbol{\theta}_{S-\boldsymbol{\nu}\cdot\boldsymbol{Y}}. \tag{26}$$

Written in this way, it is easily seen that hedging in two steps and hedging in a single step lead to the same global hedge. Indeed, first choosing a hedge $\boldsymbol{\nu}$ and then applying the hedger $\boldsymbol{\theta}$ to the remaining loss $S - \boldsymbol{\nu} \cdot \boldsymbol{Y}$ leads to the same overall hedge as immediately applying the hedger $\boldsymbol{\theta}$ on S.

The condition (25) in the definition of an actuarial hedger means that any orthogonal claim $S^\perp$ is hedged by an investment of amount $\pi\left[S^\perp\right]$ in zero-coupon bonds.

In the following lemma, we summarize some properties of hedgers that will be used hereafter. The proofs are straightforward and therefore omitted.

Lemma 1 *Consider the hedger $\boldsymbol{\theta} : \mathcal{C} \to \Theta$ with underlying actuarial valuation π.*
(1) For any MC hedger, any scalar a and any claim S, one has that

$$\boldsymbol{\theta}_{S+a} = \boldsymbol{\theta}_S + \left(e^{-r}\ a, 0, \ldots, 0\right). \tag{27}$$

(2) For any actuarial hedger, any scalar a and any orthogonal claim $S^\perp$, one has that

$$\boldsymbol{\theta}_{S^\perp+a} = \left(\pi\left[S^\perp\right] + e^{-r}a, 0, \ldots, 0\right). \tag{28}$$

(3) For any fair hedger, any hedgeable claim $\boldsymbol{\nu} \cdot \mathbf{Y}$ and any orthogonal claim $S^{\perp}$, one has that

$$\boldsymbol{\theta}_{S^{\perp}+\boldsymbol{\nu}\cdot\mathbf{Y}} = \left(\pi\left[S^{\perp}\right], 0, \ldots, 0\right) + \boldsymbol{\nu}. \tag{29}$$

As a special case of (28), we find for any actuarial hedger $\boldsymbol{\theta}_a$ that any scalar a is hedged by

$$\boldsymbol{\theta}_a = \left(e^{-r}\ a, 0, \ldots, 0\right). \tag{30}$$

Moreover, for any fair hedger $\boldsymbol{\theta}$ it follows from (29) that a hedgeable claim $\boldsymbol{\nu} \cdot \boldsymbol{Y}$ is hedged by its underlying hedge:

$$\boldsymbol{\theta}_{\boldsymbol{\nu}\cdot\mathbf{Y}} = \boldsymbol{\nu}. \tag{31}$$

In the proofs of some forthcoming theorems, we will consider a hedge $\boldsymbol{\mu}_S$ for any claim S which is defined as the sum of another hedge $\boldsymbol{\theta}_S$ of S and an actuarial hedge of the remaining risk $S - \boldsymbol{\theta}_S \cdot \mathbf{Y}$. Some properties of such hedgers are considered in the following lemma.

Lemma 2 *For any hedger $\boldsymbol{\theta}$, valuation ρ and hedger $\boldsymbol{\mu}$ defined by*

$$\boldsymbol{\mu}_S = \boldsymbol{\theta}_S + \left(\rho\left[S - \boldsymbol{\theta}_S \cdot \mathbf{Y}\right], 0, \ldots, 0\right), \qquad \textit{for any } S \in \mathcal{C}, \tag{32}$$

the following results hold:

(a) If $\boldsymbol{\theta}$ is a MC hedger, then $\boldsymbol{\mu}$ is a MC hedger.

(b) If $\boldsymbol{\theta}$ is an actuarial hedger and ρ is an actuarial valuation, then $\boldsymbol{\mu}$ is an actuarial hedger with underlying actuarial valuation ρ.

(c) If $\boldsymbol{\theta}$ is a fair hedger and ρ is an actuarial valuation, then $\boldsymbol{\mu}$ is a fair hedger with underlying actuarial valuation ρ.

Proof: (a) Suppose that $\boldsymbol{\theta}$ is a MC hedger. For any claim S and any hedgeable claim $S^h = \nu \cdot \mathbf{Y}$, we find that

$$\begin{aligned}
\boldsymbol{\mu}_{S+S^h} &= \boldsymbol{\theta}_{S+S^h} + \left(\rho\left[S + S^h - \boldsymbol{\theta}_{S+S^h} \cdot \mathbf{Y}\right], 0, \ldots, 0\right) \\
&= \boldsymbol{\theta}_S + \boldsymbol{\nu} + \left(\rho\left[S - \boldsymbol{\theta}_S \cdot \mathbf{Y}\right], 0, \ldots, 0\right) \\
&= \boldsymbol{\mu}_S + \boldsymbol{\nu}.
\end{aligned}$$

We can conclude that $\boldsymbol{\mu}$ is a MC hedger.

(b) Next, suppose that $\boldsymbol{\theta}$ is an actuarial hedger with underlying actuarial valuation π. Further, suppose that ρ is an actuarial valuation. For any orthogonal claim $S^{\perp}$, we have

$$\begin{aligned}
\boldsymbol{\mu}_{S^{\perp}} &= \boldsymbol{\theta}_{S^{\perp}} + \left(\rho\left[S^{\perp} - \boldsymbol{\theta}_{S^{\perp}} \cdot \mathbf{Y}\right], 0, \ldots, 0\right) \\
&= \left(\pi\left[S^{\perp}\right] + \rho\left[S^{\perp} - e^{r}\ \pi\left[S^{\perp}\right]\right], 0, \ldots, 0\right) \\
&= \left(\rho\left[S^{\perp}\right], 0, \ldots, 0\right),
\end{aligned}$$

where in the last step, we used the translation invariance of ρ. We can conclude that $\boldsymbol{\mu}$ is an actuarial hedger with underlying actuarial valuation ρ.
(c) Finally, suppose that $\boldsymbol{\theta}$ is a fair hedger with underlying actuarial valuation π, while ρ is an actuarial valuation. From (a) and (b) it follows immediately that $\boldsymbol{\mu}$ is a fair hedger with underlying actuarial valuation ρ. ■

In Section 4.3 , we will consider mean-variance hedging and the related mean-variance hedger which will be shown to be a fair hedger, see Corollary 3 hereafter. The mean-variance hedger is defined as follows:

$$\boldsymbol{\theta}_S^{MV} = \arg\min_{\boldsymbol{\mu}\in\Theta} \mathbb{E}^{\mathbb{P}}\left[(S - \boldsymbol{\mu}\cdot\mathbf{Y})^2\right], \qquad \text{for any } S \in \mathcal{C}. \tag{33}$$

In the following theorem it is shown that any MC valuation can be represented as the time - 0 price of MC hedger. Similar properties hold for actuarial and fair valuations.

Theorem 1 *Consider the valuation $\rho : \mathcal{C} \to \mathbb{R}$.*
(a) ρ is a MC valuation if and only if there exists a MC hedger $\boldsymbol{\theta}^m$ such that

$$\rho[S] = \boldsymbol{\theta}_S^m \cdot \boldsymbol{y}, \qquad \textit{for any } S \in \mathcal{C}. \tag{34}$$

(b) ρ is an actuarial valuation if and only if there exists an actuarial hedger $\boldsymbol{\theta}^a$ such that

$$\rho[S] = \boldsymbol{\theta}_S^a \cdot \boldsymbol{y}, \qquad \textit{for any } S \in \mathcal{C}. \tag{35}$$

(c) ρ is a fair valuation if and only if there exists a fair hedger $\boldsymbol{\theta}^f$ such that

$$\rho[S] = \boldsymbol{\theta}_S^f \cdot \boldsymbol{y}, \qquad \textit{for any } S \in \mathcal{C}. \tag{36}$$

Proof: (a) Let ρ be a MC valuation. Consider a MC hedger $\boldsymbol{\theta}$, e.g. the mean-variance hedger defined in (33). For any claim S, we find from (12) that

$$\begin{aligned}\rho[S] &= \rho[S - \boldsymbol{\theta}_S \cdot \mathbf{Y}] + \boldsymbol{\theta}_S \cdot \boldsymbol{y} \\ &= \boldsymbol{\theta}_S^m \cdot \boldsymbol{y}\end{aligned}$$

with

$$\boldsymbol{\theta}_S^m = \boldsymbol{\theta}_S + (\rho[S - \boldsymbol{\theta}_S \cdot \mathbf{Y}], 0, \ldots, 0). \tag{37}$$

From Lemma 1 we know that $\boldsymbol{\theta}^m$ is a MC hedger.
(a') Suppose that the valuation ρ is defined by (34) for some MC hedger $\boldsymbol{\theta}^m$. For any hedgeable claim $\boldsymbol{\nu}\cdot\mathbf{Y}$, we find that

$$\begin{aligned}\rho[S + \boldsymbol{\nu}\cdot\mathbf{Y}] &= \boldsymbol{\theta}_{S+\boldsymbol{\nu}\cdot\mathbf{Y}}^m \cdot \boldsymbol{y} \\ &= (\boldsymbol{\theta}_S^m + \boldsymbol{\nu}) \cdot \boldsymbol{y} \\ &= \rho[S] + \boldsymbol{\nu}\cdot\boldsymbol{y}.\end{aligned}$$

We can conclude that ρ is a MC valuation.
(b) Let ρ be an actuarial valuation. Consider the hedger $\boldsymbol{\theta}^a$ with

$$\boldsymbol{\theta}_S^a = (\rho[S], 0, \ldots, 0),$$

for any claim S. Obviously, $\boldsymbol{\theta}^a$ is an actuarial hedger. Then we find that

$$\rho[S] = \boldsymbol{\theta}^a_S \cdot \boldsymbol{y}, \qquad \text{for any } S \in \mathcal{C}.$$

(b') Suppose that the valuation ρ is defined by (35) for some actuarial hedger $\boldsymbol{\theta}^a$ with underlying actuarial valuation π. For any orthogonal claim $S^{\perp}$, we have

$$\rho\left[S^{\perp}\right] = \boldsymbol{\theta}^a_{S^{\perp}} \cdot \boldsymbol{y} = \pi\left[S^{\perp}\right].$$

We can conclude that the valuation ρ is actuarial.

(c) Let ρ be a fair valuation. Consider a fair hedger $\boldsymbol{\theta}$, e.g. the mean-variance hedger, with underlying actuarial valuation π. From (a) we know that for any claim S, $\rho[S]$ can be expressed as

$$\rho[S] = \boldsymbol{\theta}^m_S \cdot \boldsymbol{y},$$

with the MC hedger $\boldsymbol{\theta}^m$ given by (37). Furthermore, for any orthogonal claim $S^{\perp}$, we find that

$$\begin{aligned}
\boldsymbol{\theta}^m_{S^{\perp}} &= \boldsymbol{\theta}_{S^{\perp}} + \left(\rho\left[S^{\perp} - \boldsymbol{\theta}_{S^{\perp}} \cdot \mathbf{Y}\right], 0, \dots, 0\right) \\
&= \left(\pi\left[S^{\perp}\right], 0, \dots, 0\right) + \left(\rho\left[S^{\perp} - e^r\, \pi\left[S^{\perp}\right]\right], 0, \dots, 0\right) \\
&= \left(\rho\left[S^{\perp}\right], 0, \dots, 0\right).
\end{aligned}$$

As ρ is an actuarial valuation, we can conclude that the hedger $\boldsymbol{\theta}^m$ is not only market-consistent but also actuarial and hence, a fair hedger.

(c') Suppose that the valuation ρ is defined by (36) for some fair hedger $\boldsymbol{\theta}^f$. From (a) and (b) we can conclude that the valuation ρ is market-consistent, actuarial, and hence, fair. ■

From Theorem 1, we know that any fair value $\rho[S]$ can be considered as the time - 0 price of a fair hedge:

$$\rho[S] = e^{-r}\, \mathbb{E}^{\mathbb{Q}}\left[\boldsymbol{\theta}^f_S \cdot \mathbf{Y}\right],$$

where $\mathbb{Q}$ is an EMM and $\boldsymbol{\theta}^f_S$ is a fair hedger. Remark that this result is mainly of a theoretical nature, and not really useful in practice, as the fair hedge $\boldsymbol{\theta}^f_S$ is only implicitely known, see (37).

Corollary 1 *Consider the fair valuation characterized by*

$$\rho[S] = \boldsymbol{\theta}_S \cdot \boldsymbol{y}, \qquad \text{for any } S \in \mathcal{C}, \tag{38}$$

where $\boldsymbol{\theta}$ is a fair hedger with underlying actuarial valuation π. Furthermore, consider the claim $S^h \times S^{\perp}$, where $S^h = \boldsymbol{\nu} \cdot \mathbf{Y} \in C^h$ and $S^{\perp} \in C^{\perp}$. In case

$$\boldsymbol{\theta}_{S^h \times S^{\perp}} = \boldsymbol{\nu} \times e^r\, \pi\left[S^{\perp}\right] \tag{39}$$

holds, the fair value $\rho\left[S^h \times S^{\perp}\right]$ can be expressed as

$$\rho\left[S^h \times S^{\perp}\right] = \mathbb{E}^{\mathbb{Q}}\left[S^h\right] \times \pi\left[S^{\perp}\right], \tag{40}$$

for any EMM $\mathbb{Q}$.

From Theorem 1, we know that any fair valuation ρ can always be expressed as in (38). Furthermore, it is easy to verify that the condition (39) is always satisfied when $S^{\perp} = \mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right]$. In this case, we have that $\pi\left[S^{\perp}\right] = e^{-r}\ \mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right]$ and (40) reduces to the well-known Brennan & Schwartz - formula (21). In this sense, Corollary 1 is a generalization of the Brennan & Schwartz result. As we will see in a further section, the assumption (39) and hence, the relation (40) is satisfied e.g. in case $\boldsymbol{\theta}$ is the mean-variance hedger.

4 Hedge-based valuations

In this section, we present and investigate a class of fair valuations, the members of which we will call *hedge-based valuations*. We show that the classes of fair and hedge-based valuations are identical.

4.1 The general class of hedge-based valuations

In order to determine a hedge-based value of S, one first splits this claim into a hedgeable claim, which (partially) replicates S, and a remaining claim. The value of the claim S is then defined as the sum of the financial price of the hedgeable claim and the value of the remaining claim, determined according to a pre-specified actuarial valuation.

Definition 11 (Hedge-based valuation) *The valuation $\rho : \mathcal{C} \to \mathbb{R}$ is a hedge-based valuation (HB valuation) if for any claim S, the value $\rho[S]$ is determined by*

$$\rho[S] = \boldsymbol{\theta}_S \cdot \boldsymbol{y} + \pi[S - \boldsymbol{\theta}_S \cdot \mathbf{Y}]. \tag{41}$$

where $\boldsymbol{\theta}$ is a fair hedger and π is an actuarial valuation.

For any claim S, we call $\rho[S]$ the hedge-based value of S. It is easy to verify that the mapping ρ defined in (41) is normalized and translation invariant, and hence, a valuation as defined above.

From the definition above, we find that any HB valuation ρ reduces to an actuarial valuation for orthogonal claims:

$$\rho[S^{\perp}] = \pi[S^{\perp}], \qquad \text{for any } S^{\perp} \in \mathcal{C}^{\perp}.$$

Sufficient conditions for positive homogeneity and subadditivity of hedge-based valuations are considered in the next theorem.

Theorem 2 *For any HB valuation ρ with fair hedger $\boldsymbol{\theta}$ and actuarial valuation π, the following properties hold:*
(1) If $\boldsymbol{\theta}$ and π are positive homogeneous, then ρ is positive homogeneous:

$$\rho\left[a\ S\right] = a\ \rho\left[S\right], \qquad \text{for any } a > 0 \text{ and } S \in \mathcal{C}. \tag{42}$$

(2) If $\boldsymbol{\theta}$ is additive and π is subadditive, then ρ is subadditive:

$$\rho\left[S_1 + S_2\right] \leq \rho\left[S_1\right] + \rho\left[S_2\right], \qquad \text{for any } S_1, S_2 \in \mathcal{C}. \tag{43}$$

The proof of the theorem is straightforward.

In the following theorem, it is proven that the class of hedge-based valuations is equal to the class of fair valuations.

Theorem 3 *A mapping $\rho : \mathcal{C} \to \mathbb{R}$ is a HB valuation if and only if it is a fair valuation.*

Proof:
(a) Consider the HB valuation ρ defined in (41). For any claim S, we can rewrite $\rho[S]$ as

$$\rho[S] = \boldsymbol{\mu}_S \cdot \boldsymbol{y}$$

with

$$\boldsymbol{\mu}_S = \boldsymbol{\theta}_S + (\pi[S - \boldsymbol{\theta}_S \cdot \mathbf{Y}], 0, \ldots, 0). \tag{44}$$

From Lemma 2, it follows that $\boldsymbol{\mu}$ is a fair hedger with underlying actuarial valuation π. Theorem 1 leads then to the conclusion that ρ is a fair valuation.
(b) Consider the fair valuation ρ. From Theorem 1, we know that there exists a fair hedger $\boldsymbol{\theta}^f$ such that $\rho[S] = \boldsymbol{\theta}_S^f \cdot \boldsymbol{y}$ for any claim S. Define the valuation ρ' by

$$\rho'[S] = \boldsymbol{\theta}_S^f \cdot \mathbf{y} + \rho[S - \boldsymbol{\theta}_S^f \cdot \mathbf{Y}] \tag{45}$$

Obviously, ρ' is a HB valuation. Moreover, it is easy to verify that

$$\rho[S - \boldsymbol{\theta}_S^f \cdot \mathbf{Y}] = 0.$$

We can conclude that $\rho \equiv \rho'$, and hence, ρ is indeed a HB valuation. ■

One could define a broader class of HB valuations by requiring that the hedger $\boldsymbol{\theta}$ in (41) is a market-consistent hedger and π is an actuarial valuation. In this case the hedger $\boldsymbol{\mu}$ defined in (44) is market-consistent, but not necessarily fair, implying that such a generalized HB valuation is still market-consistent but not necessarily fair anymore.

4.2 Convex hedge-based valuations

We start this subsection by introducing a class of hedgers, which we will baptize convex hedgers.

Definition 12 (Convex hedger) *Consider a strictly convex non-negative function u with $u(0) = 0$. The hedger $\boldsymbol{\theta}^u$ with*

$$\boldsymbol{\theta}_S^u = \arg\min_{\boldsymbol{\mu} \in \Theta} \mathbb{E}^{\mathbb{P}}[u(S - \boldsymbol{\mu} \cdot \mathbf{Y})], \qquad \textit{for any } S \in \mathcal{C}, \tag{46}$$

is called a convex hedger (with convex function u).

The convex hedger $\boldsymbol{\theta}^u : \mathcal{C} \to \Theta$ attaches the hedge $\boldsymbol{\theta}_S^u$ to any claim S, such that the claim and the time-1 value of the hedge are 'close to each other' in the sense that the $\mathbb{P}$-expectation of the u-value of their difference is minimized. The choice of the convex function u determines how severe deviations are punished.

Theorem 4 *The convex hedger $\boldsymbol{\theta}^u$ is a fair hedger with underlying actuarial valuation π^u satisfying*

$$\pi^u\left[S^{\perp}\right] = \arg\min_{s\in\mathbb{R}} \mathbb{E}^{\mathbb{P}}\left[u\left(S^{\perp} - e^r\ s\right)\right], \qquad \textit{for any } S^{\perp}\in\mathcal{C}^{\perp}. \tag{47}$$

Proof: Consider the convex hedger $\boldsymbol{\theta}^u$ defined in (46). We have to prove that $\boldsymbol{\theta}^u$ satisfies the conditions (24) and (25) of the definition of a fair hedger.
(a) For any hedgeable claim $S^h = \boldsymbol{\nu}\cdot\mathbf{Y}$, we have that

$$\begin{aligned}\boldsymbol{\theta}^u_S &= \arg\min_{\boldsymbol{\mu}\in\Theta} \mathbb{E}^{\mathbb{P}}\left[u\left(\left(S - S^h\right) - (\boldsymbol{\mu}-\boldsymbol{\nu})\cdot\mathbf{Y}\right)\right] \\ &= \boldsymbol{\nu} + \arg\min_{\boldsymbol{\mu}'\in\Theta} \mathbb{E}^{\mathbb{P}}\left[u\left(\left(S - S^h\right) - \boldsymbol{\mu}'\cdot\mathbf{Y}\right)\right] \\ &= \boldsymbol{\nu} + \boldsymbol{\theta}^u_{S-S^h},\end{aligned}$$

which means that the condition (24) is satisfied.
(b) Consider the orthogonal claim $S^{\perp}\in\mathcal{C}^{\perp}$. Taking into account the independence of $S^{\perp}$ and $\mathbf{Y}$ as well as Jensen's inequality, we find for any trading strategy $\boldsymbol{\nu}\in\Theta$ that

$$\mathbb{E}^{\mathbb{P}}\left[u\left(S^{\perp} - \boldsymbol{\nu}\cdot\mathbf{Y}\right) \mid S^{\perp}\right] \geq u\left(S^{\perp} - \boldsymbol{\nu}\cdot\mathbb{E}^{\mathbb{P}}\left[\mathbf{Y}\right]\right).$$

Taking expectations on both sides leads to

$$\mathbb{E}^{\mathbb{P}}\left[u\left(S^{\perp} - \boldsymbol{\nu}\cdot\mathbf{Y}\right)\right] \geq \mathbb{E}^{\mathbb{P}}\left[u\left(S^{\perp} - \boldsymbol{\nu}\cdot\mathbb{E}^{\mathbb{P}}\left[\mathbf{Y}\right]\right)\right] \geq \mathbb{E}^{\mathbb{P}}\left[u\left(S^{\perp} - e^r\ \pi^u\left[S^{\perp}\right]\right)\right],$$

which holds for any $\boldsymbol{\nu}\in\Theta$. Taking into account that $e^r\ \pi\left[S^{\perp}\right]$ can be rewritten as

$$e^r\ \pi\left[S^{\perp}\right] = \left(\pi^u\left[S^{\perp}\right], 0, \ldots, 0\right)\cdot\mathbf{Y},$$

with $\left(\pi^u\left[S^{\perp}\right], 0, \ldots, 0\right)$ being an element of Θ, we find that

$$\boldsymbol{\theta}^u_{S^{\perp}} = \left(\pi^u\left[S^{\perp}\right], 0, \ldots, 0\right).$$

Let us now extend the definition (47) of π^u to all $S\in\mathcal{C}$. It is easy to verify that π^u is an actuarial valuation. We can conclude that also the condition (25) is satisfied. ■

Definition 13 (Convex hedge-based valuation) *Consider the strictly convex non-negative function u with $u(0) = 0$. The valuation $\rho : \mathcal{C}\to\mathbb{R}$ defined by*

$$\rho\left[S\right] = \theta^u_S\cdot\mathbf{y} + \pi[S - \theta^u_S\cdot\mathbf{Y}],$$

with convex hedger θ^u and actuarial valuation π is called a convex hedge-based valuation (CHB valuation).

Corollary 2 *Any CHB valuation is a fair valuation.*

The proof of the corollary follows from observing that any CHB valuation is a HB valuation, implying that it is a fair valuation.

4.3 Mean-variance hedge-based valuations

One particular example of a convex hedge-based valuation arises when using the convex hedger with quadratic function $u(s) = s^2$. This hedger is called the mean-variance hedger.

Definition 14 (Mean-variance hedging) *For any $S \in \mathcal{C}$, the mean-variance hedge θ_S^{MV} (MV hedge) is the hedge for which the $\mathbb{P}$-expected quadratic hedging error is minimized:*

$$\boldsymbol{\theta}_S^{MV} = \arg\min_{\boldsymbol{\mu}\in\Theta} \mathbb{E}^{\mathbb{P}}\left[(S - \boldsymbol{\mu}\cdot\mathbf{Y})^2\right]. \tag{48}$$

For an overview on the general theory of mean-variance hedging, we refer to Schweizer (2001).

Corollary 3 *The mean-variance hedger $\theta^{MV} : \mathcal{C} \to \Theta$ is a fair hedger with underlying actuarial valuation satisfying*

$$\pi^{MV}\left[S^{\perp}\right] = e^{-r}\,\mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right], \qquad \textit{for any } S^{\perp} \in \mathcal{C}^{\perp}. \tag{49}$$

Proof: The MV hedger is a convex hedger, implying that it is a fair hedger. From (47) it follows that it has an actuarial valuation which satisfies (49). ■

In the following theorem, we present the unique solution $\boldsymbol{\theta}_S^{MV} = \left(\theta_S^{(0)}, \ldots, \theta_S^{(n)}\right)$ of the minimization problem (48), which is a standard result from least squares optimization. We use the notation $\mathbf{A}^{\intercal}$ for the transpose of a matrix $\mathbf{A}$.

Theorem 5 *The mean-variance hedge $\boldsymbol{\theta}_S^{MV}$ of $S \in \mathcal{C}$ is uniquely determined from*

$$\mathbb{E}^{\mathbb{P}}\left[\mathbf{Y}^{\intercal}\mathbf{Y}\right]\ \boldsymbol{\theta}_S^{MV} = \mathbb{E}^{\mathbb{P}}\left[S\mathbf{Y}^{\intercal}\right]. \tag{50}$$

Proof: Taking partial derivatives of the objective function in (48) leads to (50). As the market of traded assets is assumed to be non-redundant, for any $\boldsymbol{\theta} \neq \mathbf{0}$, one has that

$$\boldsymbol{\theta}\ \mathbb{E}^{\mathbb{P}}\left[\mathbf{Y}^{\intercal}\mathbf{Y}\right]\ \boldsymbol{\theta}^{\intercal} = \mathbb{E}^{\mathbb{P}}\left[(\boldsymbol{\theta}_S\cdot\mathbf{Y})^2\right] > 0.$$

We can conclude that the matrix $\mathbb{E}^{\mathbb{P}}\left[\mathbf{Y}^{\intercal}\mathbf{Y}\right]$ is positive definite and hence, non-singular. This implies that the mean-variance hedge $\boldsymbol{\theta}_S^{MV}$ is uniquely determined and follows from (50). ■

It is a straightforward exercise to show that the system of equations (50) to determine $\boldsymbol{\theta}_S^{MV} = \left(\theta_S^{(0)}, \ldots, \theta_S^{(n)}\right)$ can be transformed into

$$\begin{cases} \displaystyle\sum_{m=1}^{n} \mathrm{cov}_{km}\ \theta_S^{(m)} = \mathrm{cov}_{k,S}, & \text{for } k = 1, \ldots, n \\ \theta_S^{(0)} = e^{-r}\left(\mathbb{E}^{\mathbb{P}}[S] - \displaystyle\sum_{m=1}^{n}\mathbb{E}^{\mathbb{P}}\left[Y^{(m)}\right]\ \theta_S^{(m)}\right) \end{cases} \tag{51}$$

with

$$\mathrm{cov}_{km} = \mathrm{cov}^{\mathbb{P}}\left[Y^{(k)}, Y^{(m)}\right] \tag{52}$$

$$\mathrm{cov}_{k,S} = \mathrm{cov}^{\mathbb{P}}\left[Y^{(k)}, S\right]. \tag{53}$$

In the following theorem, we provide some well-known properties of the mean-variance hedger.

Theorem 6 *For any mean-variance hedger $\boldsymbol{\theta}^{MV}$, the following properties hold:*
(a) Any claim S and the time-1 value of its MV hedge are equal in $\mathbb{P}$-expectation:

$$\mathbb{E}^{\mathbb{P}}[S] = \mathbb{E}^{\mathbb{P}}\left[\boldsymbol{\theta}_S^{MV} \cdot \mathbf{Y}\right], \qquad \textit{for any } S \in \mathcal{C}. \tag{54}$$

(b) The MV hedger is additive:

$$\boldsymbol{\theta}_{S_1+S_2}^{MV} = \boldsymbol{\theta}_{S_1}^{MV} + \boldsymbol{\theta}_{S_2}^{MV}, \qquad \textit{for any } S_1, S_2 \in \mathcal{C}. \tag{55}$$

(c) The MV hedger is positive homogeneous:

$$\boldsymbol{\theta}_{a\times S}^{MV} = a \times \theta_S^{MV}, \qquad \textit{for any scalar } a > 0 \textit{ and any } S \in \mathcal{C}. \tag{56}$$

(d) The MV hedge of the product of a hedgeable and an orthogonal claim:

$$\boldsymbol{\theta}_{S^h\times S^\perp}^{MV} = \nu \times \mathbb{E}^{\mathbb{P}}\left[S^\perp\right], \qquad \textit{for any } S^h = \boldsymbol{\nu} \cdot \mathbf{Y} \in \mathcal{C}^h \textit{ and } S^\perp \in \mathcal{C}^\perp. \tag{57}$$

Proof: The expression (54) follows immediately from the expression for $\theta_S^{(0)}$ in (51). The other expressions are easy to prove with the help of Theorem 5. ■

Based on the mean-variance hedgers introduced above, we can define mean-variance hedge-based valuations.

Definition 15 (Mean-variance hedge-based valuation) *The valuation $\rho : \mathcal{C} \to \mathbb{R}$ where for any claim S, $\rho[S]$ is determined by*

$$\rho[S] = \boldsymbol{\theta}_S^{MV} \cdot \mathbf{y} + \pi\left[S - \boldsymbol{\theta}_S^{MV} \cdot \mathbf{Y}\right], \tag{58}$$

with $\boldsymbol{\theta}^{MV}$ the mean-variance hedger and π an actuarial valuation, is called a mean-variance hedge-based valuation (MVHB valuation).

As any MVHB valuation is a CHB valuation, we immediately find the following result.

Corollary 4 *Any MVHB valuation is a fair valuation.*

Combining Theorems 2 and 6 leads to the following result.

Theorem 7 *For any MVHB valuation ρ with underlying actuarial valuation π, the following properties hold:*
(1) If π is positive homogeneous, then ρ is positive homogeneous.
(2) If π is subadditive, then ρ is subadditive.

In the following subsection, we illustrate the calculation of MVHB valuations with two examples.

4.4 Examples

Example 3

(a) Consider the financial-actuarial world in which a zero-coupon bond and a stock are traded. The current price of the zero-coupon bond equals $y^{(0)} = 1$, while its time-1 price is given by $Y^{(0)} = 1$. The stock trades at current price $y^{(1)} = 1/2$, whereas its value at time 1, notation $Y^{(1)}$, is either 0 or 1. In this world, we also observe a non-traded survival index. Its time-1 value $\mathcal{I}$ is either 0 (if few people of a given population survive) or 1 (in case many of them survive).

We model this financial-actuarial world by the probability space $\left(\Omega, 2^{\Omega}, \mathbb{P}\right)$, with the universe Ω given by

$$\Omega = \{(0,0), (0,1), (1,0)\,(1,1)\},$$

where each element denotes a possible scenario. The first component of any couple corresponds to a possible value of the stock price $Y^{(1)}$ at time 1, while the second component is a possible value of the survival index $\mathcal{I}$ at time 1. Suppose that the real-world probability measure $\mathbb{P}$ is characterized by

$$p_{00} = \frac{1}{6},\ p_{10} = \frac{2}{6},\ p_{01} = \frac{1}{6} \text{ and } p_{11} = \frac{2}{6},$$

where each p_{ij} stands for $\mathbb{P}\left[(i,j)\right]$. One can easily verify that the time-1 values $Y^{(1)}$ and $\mathcal{I}$ of the stock and the survival index are mutually independent under the physical measure $\mathbb{P}$.

Let us now consider the valuation of the following non-traded claim:

$$S = \left(1 - Y^{(1)}\right) \times (1 - \mathcal{I}). \tag{59}$$

The MV hedge of S is given by

$$\boldsymbol{\theta}_S^{MV} = \arg\min_{\boldsymbol{\mu}\in\Theta} \mathbb{E}^{\mathbb{P}}\left[\left(S - \mu^{(0)} - \mu^{(1)} Y^{(1)}\right)^2\right] = \left(\frac{1}{2}, -\frac{1}{2}\right).$$

The MVHB value (58) of S is then equal to

$$\rho\left[S\right] = \frac{1}{4} + \pi\left[S - \frac{1}{2} + \frac{1}{2} Y^{(1)}\right].$$

Suppose that the actuarial valuation π is a cost-of-capital principle:

$$\pi\left[X\right] = \mathbb{E}^{\mathbb{P}}\left[X\right] + 0.06\left(\mathrm{VaR}_{0.995}\left[X\right] - \mathbb{E}^{\mathbb{P}}\left[X\right]\right), \qquad \text{for any } X \in \mathcal{C}. \tag{60}$$

As $\mathbb{E}^{\mathbb{P}}\left[S\right] = \mathbb{E}^{\mathbb{P}}\left[\frac{1}{2} - \frac{1}{2} Y^{(1)}\right]$ and $\mathrm{VaR}_{0.995}\left[S - \frac{1}{2} + \frac{1}{2} Y^{(1)}\right] = 1/2$, we find that

$$\rho\left[S\right] = \frac{7}{25}.$$

(b) Next, we consider a market where in adition to the zero-coupon bond and the stock, also the survival index $\mathcal{I}$ is traded, with current price $y^{(2)} = \frac{2}{3}$ and value at time 1 given by $Y^{(2)} = \mathcal{I}$. The MV hedge of S is now given by

$$\boldsymbol{\theta}_S^{MV} = \arg\min_{\boldsymbol{\mu}\in\Theta} \mathbb{E}^{\mathbb{P}} \left[\left(S - \mu^{(0)} - \mu^{(1)} Y^{(1)} - \mu^{(2)} Y^{(2)} \right)^2 \right] = \left(\frac{2}{3}, -\frac{1}{2}, -\frac{1}{3} \right),$$

while the MVHB valuation (58) of S takes the form

$$\rho\left[S\right] = \frac{7}{36} + \pi \left[S - \frac{2}{3} + \frac{1}{2} Y^{(1)} + \frac{1}{3} Y^{(2)} \right].$$

In case the actuarial valuation π is given by the cost-of-capital principle (60), taking into account that $\mathrm{VaR}_{0.995}\left[S - \frac{2}{3} + \frac{1}{2}Y^{(1)} + \frac{1}{3}Y^{(2)}\right] = 1/3$, we find that

$$\rho\left[S\right] = \frac{193}{900}.$$

(c) Let us now assume that, apart from the zero-coupon bond, the stock and the survival index $\mathcal{I}$, also the call option with current price $y^{(3)} = \frac{1}{6}$ and payoff at time 1 given by

$$Y^{(3)} = Y^{(2)} \times (Y^{(1)} - 0.5)_+ \tag{61}$$

is traded. The MV hedge of S now equals

$$\boldsymbol{\theta}_S^{MV} = \arg\min_{\boldsymbol{\mu}\in\Theta} \mathbb{E}^{\mathbb{P}} \left[\left(S - \mu^{(0)} - \mu^{(1)} Y^{(1)} - \mu^{(2)} Y^{(2)} - \mu^{(3)} Y^{(3)} \right)^2 \right] = (1, -1, -1, 2).$$

The claim S is now perfectly hedged by its MV hedge:

$$S = Y^{(0)} - Y^{(1)} - Y^{(2)} + 2Y^{(3)}.$$

This is due to the fact that the introduction of the call option leads to a complete market, see forthcoming Example 5. In this case, the MVHB value (58) of S is given by the price of the replicating portfolio:

$$\rho\left[S\right] = \frac{1}{6}.$$

In this example, the fair value of S decreases by introducing additional traded assets. Notice however that this is not always necessarily the case. ◀

Example 4

(a) Consider a national population of N members. For member i, we introduce the Bernoulli r.v. I_i, which equals 0 if i dies in the coming year, while it equals 1 in the other case. We have that $\mathbb{P}\left[I_i = 1\right] = 1 - \mathbb{P}\left[I_i = 0\right] = p_i$, $i = 1, 2, \ldots, N$. The 'national survival index' I is given by

$$I = I_1 + I_2 + \ldots + I_N.$$

Next, we consider an insured population, consisting of M members, with J_i, $i = 1, 2, \ldots, M$, the Bernoulli r.v. which equals 1 in case insured i survives and 0 otherwise. We introduce the following notations: $\mathbb{P}\left[J_i = 1\right] = 1 - \mathbb{P}\left[J_i = 0\right] = p_i'$, $i = 1, 2, \ldots, M$. Notice that the insured population is not necessarily a subset of the national population. The insurance claim at the end of the year is given by

$$S = J_1 + J_2 + \ldots + J_M. \tag{62}$$

Suppose the financial market consists of 3 traded assets. The zero-coupon bond has current value $y^{(0)} = 1$, while its value at time 1 is given by $Y^{(0)} = e^r$. The second traded asset is a stock with current price $y^{(1)}$ and payoff at time 1 given by $Y^{(1)}$, which is either 0 or 1. Finally, also the national survival index is traded. Its current value is $y^{(2)}$, while its payoff at time 1 is given by $Y^{(2)} = I$.
We model this financial-actuarial world by the probability space $(\Omega, \mathcal{G}, \mathbb{P})$, with

$$\Omega = \{(x, y, z) \mid x = 0, 1;\ y = 0, 1..., N \text{ and } z = 0, 1, ..., M\},$$

where any triplet (x, y, z) describes a possible outcome of stock $Y^{(1)}$, the national survival index I and the insurance claim S, respectively. Throughout this example, we assume that mortality is independent of the stock price evolution. To be more precise, $Y^{(1)}$ and (I, S) are assumed to be mutually independent under the physical probability measure $\mathbb{P}$. From (51) with $n = 2$, it follows that the mean-variance hedge $\boldsymbol{\theta}_S^{MV} = \left(\theta_S^{(0)}, \theta_S^{(1)}, \theta_S^{(2)}\right)$ of the insurance claim S is given by

$$\begin{cases} \theta_S^{(0)} = e^{-r}\left(\mathbb{E}^{\mathbb{P}}[S] - \mathbb{E}^{\mathbb{P}}\left[Y^{(2)}\right] \frac{\text{cov}^{\mathbb{P}}\left[Y^{(2)}, S\right]}{\text{var}^{\mathbb{P}}\left[Y^{(2)}\right]}\right) \\ \theta_S^{(1)} = 0 \\ \theta_S^{(2)} = \frac{\text{cov}^{\mathbb{P}}\left[Y^{(2)}, S\right]}{\text{var}^{\mathbb{P}}\left[Y^{(2)}\right]}. \end{cases} \tag{63}$$

This MV hedging strategy for S does not contain an investment in the stock, due to its assumed independence with mortality. A higher correlation between the insurance claim and the national survival index leads, ceteris paribus, to a higher investment in the national survival index and a lower investment in zero coupon bonds.
(b) From here on, we assume that the insured population is a subset of the national population. More specifically, we assume that $M \leq N$ and $J_i = I_i$ for $i = 1, 2, \ldots, M$. Furthermore, all I_i are assumed to be i.i.d. with $\mathbb{P}[I_i = 1] = 1 - \mathbb{P}[I_i = 0] = p$. In this case we find that

$$\text{cov}^{\mathbb{P}}\left[Y^{(2)}, S\right] = \text{var}^{\mathbb{P}}[S].$$

Taking into account the i.i.d. assumption of the Bernoulli variables, one has that $\text{var}^{\mathbb{P}}[S] = Mp(1-p)$, $\text{var}^{\mathbb{P}}\left[Y^{(2)}\right] = Np(1-p)$. These observations lead to the following MV hedge for S:

$$\begin{cases} \theta^{(0)} = 0 \\ \theta^{(1)} = 0 \\ \theta^{(2)} = \frac{M}{N}, \end{cases} \tag{64}$$

which corresponds with an investment in the national survival index only. The MVHB value (58) of S is then given by

$$\rho[S] = \frac{M}{N} y^{(2)} + \pi\left[S - \frac{M}{N} Y^{(2)}\right]$$

Suppose now that the actuarial valuation π is the standard-deviation principle:

$$\pi[X] = \mathbb{E}^{\mathbb{P}}[X] + \beta\sqrt{\mathrm{var}[X]}, \qquad \text{for any } X \in \mathcal{C},$$

for some $\beta \geq 0$. In this case, we find that the MVHB value of S is given by

$$\rho[S] = \frac{M}{N} y^{(2)} + \beta\sqrt{\frac{M}{N}(N-M)\,p(1-p)}. \tag{65}$$

Obviously, when $M = N$, the insurance claim S is fully hedgeable, and we find that $\rho[S]$ is equal to the time-0 price of the national survival index.

◄

5 Two-step valuations

5.1 Conditional valuations and two-step valuations

Pelsser and Stadje (2014) propose the class of two-step valuations. Hereafter, we will describe a class of valuations with members that are closely related, but slightly different from the two-step valuations introduced by these authors. Hereafter, a financial claim has to be understood as a claim that can be expressed in the form $f(\mathbf{Y})$, for some measurable function function f. Hence, a financial claim is a random variable defined on the measurable space $(\Omega, \mathcal{F}^{\mathbf{Y}})$, where $\mathcal{F}^{\mathbf{Y}} \subseteq \mathcal{G}$ is the sigma-algebra generated by the asset prices $\mathbf{Y}$.

Definition 16 (Conditional valuation) *A conditional valuation is a mapping $\pi_{\mathbf{Y}}$ attaching a financial claim to any claim S:*

$$S \to \pi_{\mathbf{Y}}[S]$$

such that
(1) $\pi_{\mathbf{Y}}$ is conditionally translation invariant:

$$\pi_{\mathbf{Y}}\left[S + S^h\right] = \pi_{\mathbf{Y}}[S] + e^{-r}\, S^h, \quad \text{for any } S \in \mathcal{C} \text{ and } S^h \in \mathcal{C}^h.$$

(2) $\pi_{\mathbf{Y}}$ reduces to an actuarial valuation on $\mathcal{C}^{\perp}$:

$$\pi_{\mathbf{Y}}\left[S^{\perp}\right] = \pi\left[S^{\perp}\right], \quad \text{for any } S^{\perp} \in \mathcal{C}^{\perp},$$

for some actuarial valuation π.

A conditional actuarial valuation is a mapping from the set of claims defined on $(\Omega, \mathcal{G})$ to the set of claims defined on $\left(\Omega, \mathcal{F}^{\mathbf{Y}}\right)$. Notice that the $\mathcal{F}^{\mathbf{Y}}$- measurable claim $\pi_{\mathbf{Y}}[S]$ may be hedgeable or not. Our definition of a conditional valuation is closely related but slightly different from the one proposed in Pelsser and Stadje (2014).

A first example of a conditional valuation is the *conditional standard deviation principle*:

$$\pi_{\mathbf{Y}}[S] = e^{-r}\left(\mathbb{E}^{\mathbb{P}}[S \mid \mathbf{Y}] + \alpha\sqrt{\mathrm{var}^{\mathbb{P}}[S \mid \mathbf{Y}]}\right), \qquad \text{for any } S \in \mathcal{C}, \tag{66}$$

where α is a non-negative real number.

A second example of a conditional valuation is the *conditional cost-of-capital principle*:

$$\pi_{\mathbf{Y}}[S] = e^{-r}\left(\mathbb{E}^{\mathbb{P}}[S \mid \mathbf{Y}] + i\ \left(\mathrm{VaR}_p[S \mid \mathbf{Y}] - \mathbb{E}^{\mathbb{P}}[S \mid \mathbf{Y}]\right)\right),, \qquad \text{for any } S \in \mathcal{C}, \tag{67}$$

for a given probability level p and perunage i, and where $\mathrm{VaR}_p[S \mid \mathbf{Y}]$ is the Value-at-Risk of S at confidence level p, conditional on the available information concerning the asset prices at time 1.

As a third example of a conditional valuation, consider a fair hedger $\boldsymbol{\theta}^f$ and let $\pi_{\mathbf{Y}}$ be defined by

$$\pi_{\mathbf{Y}}[S] = e^{-r}\ \boldsymbol{\theta}_S^f \cdot \mathbf{Y}. \tag{68}$$

It is a straightforward exercise to prove that both (66) and (68) are conditional valuations. As for any orthogonal claims $S^{\perp}$ we have that $\pi_{\mathbf{Y}}\left[S^{\perp}\right]$ does not depend on $\mathbf{Y}$, we will often denote $\pi_{\mathbf{Y}}\left[S^{\perp}\right]$ by $\pi\left[S^{\perp}\right]$. For any conditional valuation, one has that

$$\pi_{\mathbf{Y}}[a] = e^{-r}\ a$$

holds for any scalar a.

Definition 17 (Two-step valuation) *A mapping $\rho : \mathcal{C} \to \mathbb{R}$ is a two-step valuation (TS valuation) if there exists a conditional valuation $\pi_{\mathbf{Y}}$ and an EMM $\mathbb{Q}$ such that for any claim S, $\rho[S]$ is determined by*

$$\rho[S] = \mathbb{E}^{\mathbb{Q}}[\pi_{\mathbf{Y}}[S]]. \tag{69}$$

One can easily verify that the mapping defined in (69) is normalized and translation invariant, implying that a TS valuation is indeed a valuation as defined above.

The two-step valuation is characterized by a conditional actuarial valuation $\pi_{\mathbf{Y}}$ and an EMM $\mathbb{Q}$. For any claim S, $\rho[S]$ is called the two-step value (TS value) of S. It is determined by first applying the conditional actuarial valuation $\pi_{\mathbf{Y}}$ to S, and then determining the market price of the financial claim $e^r\ \pi_{\mathbf{Y}}[S]$, based on a given pricing measure $\mathbb{Q}$.

Pelsser and Stadje (2014) assume that the financial market of the $(n+1)$ traded assets is complete in $\left(\Omega, \mathcal{F}^{\mathbf{Y}}\right)$. Equivalenty stated, they assume that any financial claim $f(\mathbf{Y})$ is hedgeable. This completeness assumption implies that any claim $\pi_{\mathbf{Y}}[S]$ is hedgeable, and hence, its market value is uniquely determined. The completeness condition means that there exist a mapping $\theta^{TS} : \mathcal{C} \rightarrow \Theta$ such that

$$\boldsymbol{\theta}_S^{TS} \cdot \mathbf{Y} = e^r \ \pi_{\mathbf{Y}} [S], \qquad \text{for any } S \in \mathcal{C}. \tag{70}$$

We call $\boldsymbol{\theta}^{TS}$ the *two-step hedger of the two-step valuation* ρ. Due to the non-redundancy assumption, the time-1 value (70) of $\boldsymbol{\theta}_S^{TS}$ uniquely determines $\boldsymbol{\theta}_S^{TS}$. It is straightforward to prove that $\boldsymbol{\theta}^{TS}$ is a fair hedger with

$$\boldsymbol{\theta}_{S^\perp}^{TS} = \left(\pi \left[S^\perp\right], 0, \ldots, 0\right), \qquad \text{for any } S^\perp \in \mathcal{C}^\perp. \tag{71}$$

The TS value $\rho[S]$ of S can be expressed as

$$\rho[S] = e^{-r} \ \mathbb{E}^{\mathbb{Q}} \left[\boldsymbol{\theta}_S^{TS} \cdot \mathbf{Y}\right] = \boldsymbol{\theta}_S^{TS} \cdot \mathbf{y}, \tag{72}$$

which does not depend on the particular choice of the pricing measure $\mathbb{Q}$.

Hereafter, we will not make the completeness assumption, which implies that we have to choose a particular measure $\mathbb{Q}$ in the set of all feasible pricing measures and hence $\rho[S]$ might depend on this choice.

In the special case there is no financial market, except the risk-free bank account, any claim S is an orthogonal claim, and the two-step valuation reduces to an actuarial valuation:

$$\rho[S] = \pi[S].$$

As an example of a two-step valuation, consider the *two-step standard deviation valuation*, where the value of any claim S is determined by

$$\rho[S] = e^{-r} \ \mathbb{E}^{\mathbb{Q}} \left[\mathbb{E}^{\mathbb{P}} [S \mid \mathbf{Y}] + \beta \ \sqrt{\mathrm{Var}^{\mathbb{P}} [S \mid \mathbf{Y}]}\right]. \tag{73}$$

This means that $\rho[S]$ is determined as the financial market price of the financial claim that arises from applying the conditional standard deviation principle on the claim S, given the time-1 prices of traded assets.

A second example of a two-step valuation is the *two-step cost-of-capital valuation*:

$$\rho[S] = e^{-r} \ \mathbb{E}^{\mathbb{Q}} \left[\mathbb{E}^{\mathbb{P}} [S \mid \mathbf{Y}] + i \ \left(\mathrm{VaR}_p [S \mid \mathbf{Y}] - \mathbb{E}^{\mathbb{P}} [S \mid \mathbf{Y}]\right)\right] \tag{74}$$

In the following theorem, we prove that the class of two-step valuations is identical to the class of fair valuations.

Theorem 8 *A mapping $\rho : \mathcal{C} \rightarrow \mathbb{R}$ is a TS valuation if and only if it is a fair valuation.*

Proof: (a) Consider the TS valuation ρ with $\rho[S] = \mathbb{E}^{\mathbb{Q}} [\pi_{\mathbf{Y}} [S]]$ for any claim S. The conditional valuation $\pi_{\mathbf{Y}}$ reduces to the actuarial valuation π on $\mathcal{C}^\perp$. Let $\boldsymbol{\theta}$ be a fair hedger defined on $\mathcal{C}$. E.g, one could choose $\boldsymbol{\theta}$ to be the mean-variance hedger. For any claim S, one can rewrite the TS value $\rho[S]$ as

$$\begin{aligned}\rho[S] &= \mathbb{E}^{\mathbb{Q}}\left[\pi_{\mathbf{Y}}\left[(S-\boldsymbol{\theta}_S\cdot\mathbf{Y})+\boldsymbol{\theta}_S\cdot\mathbf{Y}\right]\right]\\ &= \mathbb{E}^{\mathbb{Q}}\left[\pi_{\mathbf{Y}}\left[S-\boldsymbol{\theta}_S\cdot\mathbf{Y}\right]+e^{-r}\ \boldsymbol{\theta}_S\cdot\mathbf{Y}\right]\\ &= \rho\left[S-\boldsymbol{\theta}_S\cdot\mathbf{Y}\right]+\boldsymbol{\theta}_S\cdot\mathbf{y}\\ &= \boldsymbol{\mu}_S\cdot\mathbf{y},\end{aligned}$$

with $\boldsymbol{\mu}$ defined by

$$\boldsymbol{\mu}_S = \boldsymbol{\theta}_S + \left(\rho\left[S-\boldsymbol{\theta}_S\cdot\mathbf{Y}\right],0\ldots,0\right).$$

As $\rho\left[S^{\perp}\right]=\pi\left[S^{\perp}\right]$ for orthogonal claims, where π is an actuarial valuation, we immeditaly find that ρ is an actuarial valuation. From Lemma 2, it follows then that $\boldsymbol{\mu}$ is a fair hedger. From Theorem 1, we can conclude that ρ is a fair valuation.
(b) Consider the fair valuation $\rho[S]=\boldsymbol{\theta}_S^f\cdot\boldsymbol{y}$ with underlying actuarial valuation π' of the fair hedger $\boldsymbol{\theta}^f$. For any claim S, we can write $\rho[S]$ as

$$\rho[S]=\mathbb{E}^{\mathbb{Q}}\left[\pi_{\mathbf{Y}}[S]\right]$$

with

$$\pi_{\mathbf{Y}}[S]=e^{-r}\ \boldsymbol{\theta}_S^f\cdot\mathbf{Y}.$$

Obviously, $\pi_{\mathbf{Y}}$ is a conditional valuation with underlying actuarial valuation π'. We can conclude that ρ is a TS valuation. ■

Consider the two-step valuation ρ with underlying conditional actuarial valuation π. Under the assumption that π is conditionally positive homogeneous, i.e.

$$\pi\left[S^h\times S\mid\mathbf{Y}\right]=S^h\times\pi\left[S\mid\mathbf{Y}\right],\qquad\text{for any non-negative } S^h\in\mathcal{C}^h \text{ and } S\in\mathcal{C},$$

we find that the two-step valuation of $S^h\times S^{\perp}$, with $S^h\geq 0$, is given by

$$\rho\left[S^h\times S^{\perp}\right]=\mathbb{E}^{\mathbb{Q}}\left[S^h\right]\times\pi\left[S^{\perp}\right].\tag{75}$$

From this result, it follows that (75) is an intuitive generalization of the valuation principle (21), proposed by Brennan and Schwartz (1976) for claims of the form $S^h\times\mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right]$ to claims of the form $S^h\times S^{\perp}$. Both valuations coincide in case of complete diversification of the orthogonal claim, i.e. when $S^{\perp}$ is equal to $\mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right]$.

5.2 Examples

We end this section with two illustrative examples, which are the counterparts of the Examples 3 and 4 considered in Subsection 4.4.

Example 5
(a) Consider the financial-actuarial world $\left(\Omega,2^{\Omega},\mathbb{P}\right)$ as described in Example 3, with a non-traded survival index and a market of traded assets consisting of a zero-coupon

bond and a stock. Suppose we want to determine the fair value $\rho[S]$ of the non-traded hybrid claim S defined in (59) according to the two-step cost-of-capital valuation (74) with $\boldsymbol{Y} = \left(Y^{(0)}, Y^{(1)}\right)$, $r = 0$, $p = 0.995$ and $i = 0.06$. Taking into account that the conditional cost-of-capital principle is conditionally positive homogeneous and that $\mathcal{I} \in \mathcal{C}^{\perp}$, we find from (75) that

$$\rho[S] = \mathbb{E}^{\mathbb{Q}}\left[1 - Y^{(1)}\right] \times \pi[1 - \mathcal{I}] = \frac{53}{200}.$$

(b) Suppose now that, apart from the zero-coupon bond and the stock, also the survival index $\mathcal{I}$ is traded, with current price $y^{(2)} = 2/3$ and time-1 value $Y^{(2)} = \mathcal{I}$. In this case, the two-step cost-of-capital valuation (74) transforms into

$$\rho[S] = \mathbb{E}^{\mathbb{Q}}\left[\left(1 - Y^{(1)}\right) \times \left(1 - Y^{(2)}\right)\right],$$

where $\boldsymbol{Y} = \left(Y^{(0)}, Y^{(1)}, Y^{(2)}\right)$. In order to fully characterize the two-step valuation ρ, one has to choose a particular risk-neutral measure $\mathbb{Q}$ of the combined market. One can verify that in this market the set of equivalent martingale measures consists of all measures $\mathbb{Q}$ characterized by

$$q_{00} = q, \ q_{10} = \frac{1}{3} - q, \ q_{01} = \frac{1}{2} - q \text{ and } q_{11} = \frac{1}{6} + q, \tag{76}$$

for some

$$q \in \left(0, \frac{1}{3}\right). \tag{77}$$

Given that the payoff of S only differs from zero in the scenario $\left(Y^{(1)}, Y^{(2)}\right) = (0, 0)$, we find that

$$\rho[S] = q.$$

The two-step value $\rho[S]$ can take any value in $\left(0, \frac{1}{3}\right)$, depending on the choice of the EMM.

(c) Let us now assume that also the call option with current price $y^{(3)} = \frac{1}{6}$ and payoff at time 1 given by (61) is traded in the market. In this case, the set of EMM's is defined by (76) and (77), complemented with the requirement

$$y^{(3)} = \mathbb{E}^{\mathbb{Q}}\left[Y^{(3)}\right].$$

This leads to a complete market with unique EMM $\mathbb{Q}$ characterized by (76) with $q = \frac{1}{6}$. We can conclude that the fair value of S is given by

$$\rho[S] = \frac{1}{6}.$$

Notice that under this unique EMM $\mathbb{Q}$, the payoffs $Y^{(1)}$ and $Y^{(2)}$ are mutually independent. ◀

Example 6 Consider the financial-actuarial world described in Example 4(b). Suppose that the insurance claim S defined in (62) is valuated according to the two-step mean-variance valuation (73), with $\boldsymbol{Y} = \left(Y^{(0)}, Y^{(1)}, Y^{(2)}\right)$, and $\beta \geq 0$. From the assumed independence between mortality and the stock price, we find that

$$\rho\left[S\right] = e^{-r}\ \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{P}}\left[S \mid Y^{(2)}\right] + \beta\sqrt{\mathrm{var}^{\mathbb{P}}\left[S \mid Y^{(2)}\right]}\right]$$

Taking into account that

$$\mathbb{E}^{\mathbb{P}}\left[S \mid Y^{(2)}\right] = \frac{M}{N}Y^{(2)}$$

and

$$\mathrm{var}^{\mathbb{P}}\left[S \mid Y^{(2)}\right] = \frac{M(N-M)}{N(N-1)}Y^{(2)}\frac{N-Y^{(2)}}{N},$$

one has that

$$\rho\left[S\right] = \frac{M}{N}y^{(2)} + \beta\ e^{-r}\ \mathbb{E}^{\mathbb{Q}}\left[\sqrt{\frac{M(N-M)}{N(N-1)}Y^{(2)}\frac{N-Y^{(2)}}{N}}\right]. \tag{78}$$

In case the financial market is complete, the EMM $\mathbb{Q}$ is unique and hence, the two-step valuation (78) can be uniquely determined. Otherwise, the incompleteness of the market requires the choice of an EMM $\mathbb{Q}$ for the valuation of S. One can easily verify that the two-step value $\rho\left[S\right]$ of S equals the price of the 'national survival index' in case the insurance and the national populations coincide. ◀

6 Final remarks

The fair value of a hybrid claim, which is by definition neither hedgeable nor orthogonal, is in general not uniquely determined, not only due to the involvement of actuarial judgement, but at an earlier stage in the valuation process also due to the ambiguity that exists in how to combine financial and actuarial pricing.

In this paper we proposed a unified framework to combine market-consistent and actuarial valuations in a so-called *fair valuation*. A market-consistent valuation of claims in such a setting is based on an extension of cash invariance to all hedgeable claims, such that all claims are valuated in agreement with current market prices. Under a market-consistent valuation, the valuation of hedgeable claims is consistent with risk-neutral pricing based on an EMM $\mathbb{Q}$. An actuarial valuation on the other hand, is typically performed with an actuarial premium principle, based on a physical probability measure $\mathbb{P}$, chosen by the actuary. In such a setting, the problem the actuary is solving is to value the claim such that the insurer will be able to pay the observed claim amount at the end of the period, ignoring the existence of a financial market. The fair valuation combines the financial approach of a market-consistent valuation and the actuarial approach of an actuarial valuation. This fair valuation makes use of $\mathbb{P}$- and $\mathbb{Q}$-measures and, in this sense, can be considered as the right setting to value hybrid claims which typically have financial and actuarial components.

We also presented a fair valuation technique, baptized hedge-based valuation, where one first unbundles the hybrid claim in a hedgeable claim (determined from the original claim according to some well-defined hedging procedure) and the remaining claim (i.e. the original claim minus the payoff of the hedgeable claim). The fair value of the claim is then defined as the sum of the financial market price of the hedge and the actuarial value of the remaining claim.

We have shown that the set of fair valuations coincides with the set of hedge-based valuations and the set of two-step valuations. The set of two-step valuations considered in this paper are inspired from the two-step valuations of Pelsser and Stadje (2014) but are nevertheless slightly different and defined in such a way that this class of valuations is identical to the class of fair valuations (see Theorem 8). The two-step and the hedge-based approaches are only two different ways of identifying the different members of this set. The advantage of the hedge-based approach is that it leads to an explicit hedge, whereas for the two-step approach, the hedge is only determined implicitely (compare the expressions (41) and (70)). In case the market of traded assets is incomplete in $\left(\Omega, \mathcal{G}^{\mathbf{Y}}, \mathbb{P}\right)$, the hedge-based approach does not require the choice of an EMM, whereas the two-step approach does.

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