KU LEUVEN

Foundations of Quantitative Risk Measurement

Chapter 1: Expected Utility Theory[1]

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[1] Chapter 1 from ‘Managing and measuring actuarial risks’, Dhaene, J., Denuit, M., Goovaerts, M., Kaas, R. & Linders, D. (2017), To be published.

- Examples of decision making problems:
  - *Individual*: bear a risk or insure it (partially)?
  - *Insurer*: accept a risk or not?
  - *Insurer*: reinsure (part of) the accepted risks?
- Optimal choice of the decision maker depends on:
  - his initial wealth,
  - his risk appetite.
- Theories of choice under risk:
  - Expected utility theory: Cramer (1728), Bernouilli 1738), Von Neumann & Morgenstern (1947).
  - Dual theory of choice under risk: Yaari (1987), Roëll (1987), Schmeidler (1989).
- Common properties of these theories:
  - Preference relations of a decision maker are qualitative in nature,
  - but follow from comparing numerical quantities.

- Problem:
  - *A fair coin is tossed repeatedly until it lands head up. The income you receive is equal to 2^n if the first head appears on the n-th toss. How much are you willing to pay for this game?*
- Expected gain:
  - Assume that the coin is fair.
  - Probability to win the amount 2^n is $\frac{1}{2^n}$.
  - The expected gain:

$$\sum_{n=1}^{+\infty} (2^n) \times \frac{1}{2^n} = \sum_{n=1}^{+\infty} 1 = +\infty.$$

- Conclusions:
  - A decision maker will not pay $+\infty$.
  - The price to play this game will be modest.
  - The expectation is not (always) a good method to value a game.

Expected utility theory

- Classical expected utility theory:
  - Each decision maker assigns a utility $u(x)$ to any fortune of amount x.
  - Utility functions are of a subjective nature.
  - 'Reasonable' utility functions share common properties:
    - non-decreasingness,
    - decreasing marginal utility.
- Expected utility and insurance:
  - Why is an individual willing to pay a premium larger than the average expected loss?
  - Why are certain insurance covers to be preferred over others?
  - Behavior of insureds:
    - moral hazard,
    - anti-selection.

Solution of G. Cramer (1728) and D. Bernoulli (1738)

- Consider a decision maker with initial fortune w.
- He attaches a utility $u(x)$ to a fortune x.
- The price to play the game is P.
- Assume our agent wins after n throws:
  - His utility if he wins after n throws: $u\left(w-P+2^n\right)$.
  - Probability to win after n throws: $\frac{1}{2^n}$.
- Expected utility:
  - At initiation, the utility he will reach if he plays the game is unknown.
  - Expected utility:

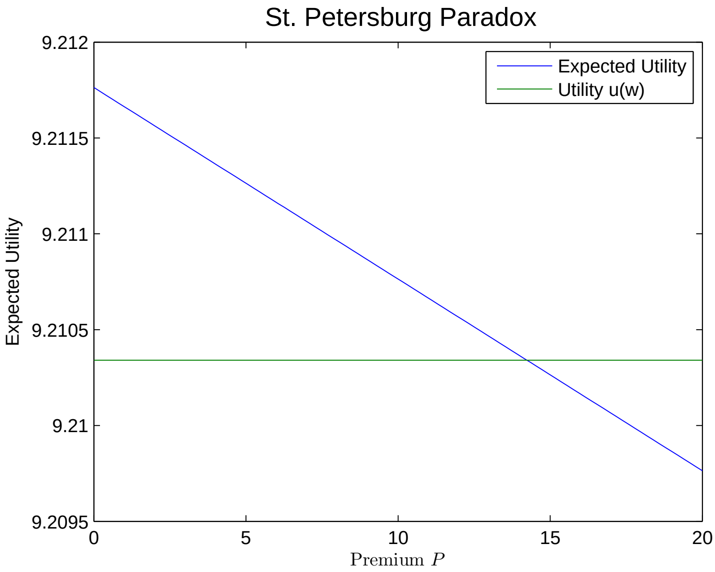
$$\sum_{n=1}^{+\infty} u\left(w-P+2^n\right)\frac{1}{2^n}.$$

Solution of G. Cramer (1728) and D. Bernoulli (1738)

- The decision maker is an *expected utility maximizer*.
- If he doesn't play the game, his utility is $u(w)$.
- He is willing to play the coin tossing game for a price P if

$$u(w) \leq \sum_{n=1}^{+\infty} u\left(w - P + 2^n\right) \frac{1}{2^n}$$

  - G. Cramer: $u(x) = \sqrt{x}$.
  - D. Bernoulli: $u(x) = \ln x$.
- Example:
  - Take $w = 10000$ and $u(x) = \ln x$.
  - Then $P = 14.2385$. (Check this using MatLab or Excel!)


St. Petersburg Paradox
Expected Utility
Utility u(w)
9.212
9.2115
9.211
9.2105
9.21
9.2095
0
5
10
15
20
Expected Utility
Premium P

Potential gains/losses

- A risk is an event solely due to the whims of fate that may or may not take place
  - and that brings about some financial loss,
  - or a financial gain.
- Examples:
  - For an insurer, a risk is a potential loss (e.g. car insurance);
  - For an investor, a risk is a potential gain (e.g. investment).
- A risk always contains uncertainty:
  - The event that may or may not take place,
  - or the severity of the consequences of its occurence,
  - or the moment of its occurrence.
- Risk vs. loss:
  - ‘Risk’ and ‘loss’ are synonyms.
- Risks are modeled by *random variables.*

1 – Random variables

R.v.'s defined on a probability space

- Consider a random experiment, defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$:
  - Ω : set with all possible outcomes;
  - $\mathcal{F}$ subsets of Ω, called events;
  - $\mathbb{P}$: probability measure:

$$\mathbb{P}[A] = \text{probability that the realization lies in the set } A \in \mathcal{F}$$

- <u>Definition:</u>
  - A random variable X defined on $(\Omega, \mathcal{F}, \mathbb{P})$ is a *function* which attaches a *real number* to each possible outcome:

$$X : \Omega \longrightarrow \mathbb{R}.$$

  - ω describes the state of a random phenomenon.
  - $X(\omega)$ is a single aspect of the state ω.

- Question: What is the probability that $X(\omega)$ lies in the interval B?
  - Probability function $\mathbb{P}$ assigns probabilities to subsets of Ω.
  - The set $X^{-1}(B)$[2]:
$$X^{-1}(B) = \{\omega | X(\omega) \in B\}$$
  - $\mathbb{P}\left[X^{-1}(B)\right]$ = probability that X takes a value in B.
  - Notation:
$$\mathbb{P}\left[X \in B\right] = \mathbb{P}\left[X^{-1}(B)\right].$$
- We assume that the probability $\mathbb{P}[X \in B]$ is known.
- *The only uncertainty when considering a future random loss is the uncertainty about its particular future outcome, not the uncertainty about its 'distribution'.*

[2] we silently assumed that X is a measurable function.

cdf of a random variable

- Cumulative distribution function (cdf) F_X of the r.v. X :

$$F_X(x) = \mathbb{P}[X \leq x], \text{ for } x \in \mathbb{R}.$$

  - F_X is non-decreasing and right continuous.
- Assume F_X is constant on $[a, b]$.
  - Probability of ending in $(a, b]$ is zero.
- Assume F_X has a jump of size $\Delta(x)$ in x :
  - $\Delta(x) = F_X(x) - F_X(x-)$.
  - $\Delta(x)$ is zero if F_X is continuous in x.
  - For all $x \in \mathbb{R}$:

$$\mathbb{P}[X = x] = \Delta(x).$$

Expectation as a Riemann-Stieltjes integral

- The average or expected value of X is denoted by $\mathbb{E}[X]$:

$$\mathbb{E}[X] = \int_{-\infty}^{+\infty} x \mathrm{d}F_X(x).$$

- If F_X has only a discrete part:

$$\mathbb{E}[X] = \sum_y y\Delta(y) = \sum_y y\mathbb{P}[X = y].$$

- If F_X has a discrete and continuous part:

$$\mathbb{E}[X] = \int_{-\infty}^{+\infty} x f_X(x)\, dx + \sum_y y\Delta(y).$$

  - $f_X(x)\mathrm{d}x =$ probability that X takes a value in the $[x, x + \mathrm{d}x]$.

- Exercise:
  - Consider a r.v. X which takes the value 0 or 1 with equal probability. Determine the cdf F_X and $\mathbb{E}[X]$.

Definition (Utility function)

A **utility function** u is a real-valued *non-decreasing* function asserting a decision maker's utility-of-wealth $u(x)$ to each possible level of wealth x.

- Decision makers have *non-negative marginal utility*: more wealth is always preferred over less wealth.
- In general, different decision makers will have different utility functions.
- We study classes of decision makers, which all share some common risk preferences

Expected utility

- Consider a decision maker having initial wealth w and facing a loss X.
- Wealth after suffering the loss X :

$$w - X.$$

- Utility level after suffering the loss X

$$u\left(w - X\right).$$

  - $u\left(w - X\right)$ is a r.v.

- The expected utility is the quantity:

$$\mathbb{E}\left[u(w - X)\right].$$

Profit-seeking decision makers

- The expected utility hypothesis:

$$\text{Prefer loss } X \text{ over loss } Y \iff \mathbb{E}\left[u(w-X)\right] \geq \mathbb{E}\left[u(w-Y)\right],$$
$$\text{Indifferent between } X \text{ and } Y \iff \mathbb{E}\left[u(w-X)\right] = \mathbb{E}\left[u(w-Y)\right].$$

  - Relations as above hold 'provided the expectations exist'.
  - The decision maker is said to be an *expected utility maximizer*.
  - Indifferent between losses with the same distribution.

- Standardized utility functions:
  - A utility function only needs to be determined up to positive linear transformations.
    - Exercise: prove this statement!
  - Standardize the utility function u:

$$u\left(x_0\right) = 0 \text{ and } u'\left(x_0\right) = 1, \text{for some } x_0 \in \mathbb{R}.$$

Transformed wealth levels

- Axiomatic framework - Von Neumann & Morgenstern (1947):
  - *Any decision maker whose behavior is in accordance with a given set of 'rational' axioms, is an expected utility maximizer.*
- The '*independence axiom*':
  - For any random losses X, Y and Z and for any Bernoulli r.v. I, independent of X, Y and Z, one has:

$$\begin{aligned} & \text{Prefer loss } X \text{ over loss } Y \\ \Rightarrow \quad & \text{Prefer loss } IX + (1-I)Z \text{ over loss } IY + (1-I)Z \end{aligned}$$

  - Example.

Definition (concave function)

A real-valued function f, defined on the interval $I \subseteq \mathbb{R}$, is **concave** on I if for any x_1, $x_2 \in I$ and any $t \in [0,1]$,

$$f\left(tx_1 + (1-t)x_2\right) \geq tf\left(x_1\right) + (1-t)f\left(x_2\right)$$

- f is **convex** on the interval I if $(-f)$ is concave on I.
- Assume f is twice differentiable:
  - f is concave $\Leftrightarrow f''(x) \leq 0$, for all $x \in I$.
  - f is convex $\Leftrightarrow f''(x) \geq 0$, for all $x \in I$.
- f is concave $\Rightarrow f$ is continuous.

Definition (Risk averse decision makers)

A decision maker is **risk averse** if his utility function u is concave on its domain.

- Risk averse decision makers have *decreasing marginal utility*.
  - Assume you gain the amount Δ.
  - Increase in utility: $u(x+\Delta) - u(x)$.
  - For risk averse decision makers, the increase in utility is a decreasing function of the wealth level x.
- Interpretation:
  - As more wealth is available, less ‘moral value’ is placed on earning an additional Euro.

Theorem (Jensen's inequality (1906))

$$f \text{ is concave } \Rightarrow \mathbb{E}\left[f(Y)\right] \leq f\left(\mathbb{E}[Y]\right)$$

- Corollary: If u is a concave utility function, then

$$\mathbb{E}\left[u(w-X)\right] \leq u\left(w-\mathbb{E}[X]\right).$$

  - Exercise: prove this inequality.
- The risk averse decision maker's attitude towards risk:
  - Prefer certainty over uncertainty with the same expectation.
- The risk averse decision maker's attitude towards wealth:
  - Decreasing marginal utility.

Expected utility and risk aversion

- <u>Definition</u>:
  A decision maker is *risk neutral* if

$$u(x) = ax + b$$

  for given constants $a > 0$ and b.

- In this case, the expected utility hypothesis coincides with comparing expected values.

- <u>The Arrow-Pratt measure of absolute risk aversion</u>:

$$r(x) = \frac{-u''(x)}{u'(x)} = -\frac{d}{dx}\ln(u'(x))$$

For any risk averse decision maker, we have that $r \geq 0$.

Expected utility and insurance

- Risk averse individual:
  - facing a loss $X \geq 0$,
  - utility function $u(x)$,
  - initial wealth w.
- Risk averse insurer:
  - accepts X for a premium P,
  - utility function U,
  - initial wealth W.
- Under what conditions is an insurance contract feasible?
  - From the viewpoint of the individual,
  - from the viewpoint of the insurer.

Expected utility and insurance

- Viewpoint of the individual:
  - He is only willing to underwrite the insurance if

$$u(w-P) \geq \mathbb{E}[u(w-X)].$$

  - There exists always a premium P^M such that

$$u\left(w-P^M\right) = \mathbb{E}[u(w-X)].$$

    - ★ u is non-decreasing.
    - ★ u is concave, hence also continuous.
    - ★ P^M is the maximum premium the insured is willing to pay.
- From Jensen's inequality:

$$P^M \geq \mathbb{E}[X].$$

  - Exericse: prove this inequality.

Expected utility and insurance

- Viewpoint of the insurer:
  - He is willing to insure the risk X at a premium P if

$$U(W) \leq \mathbb{E}\left[U(W+P-X)\right].$$

  - Minimal premium P^m he requires follows from

$$U(W) = \mathbb{E}\left[U(W+P^m-X)\right].$$

  - From Jensen's inequality:

$$P^m \geq \mathbb{E}\left[X\right].$$

    - Exercise: prove this inequality.

- Condition for an insurance contract to be feasible:

$$P^m \leq P \leq P^M$$

Expected utility and mutual exclusivity

- Definition:
  - The random vector $(X_1, X_2, \ldots, X_n)$ is said to be mutually exclusive if the following conditions hold:

$$\mathbb{P}\left[X_i \neq 0, X_j \neq 0\right] = 0, \qquad \forall i \neq j$$

- Examples of mutual exclusive couples:
  - Insurance with a franchise deductible:

$$\varphi(X) = \begin{cases} 0 \text{ if } X \leq d \\ X \text{ otherwise} \end{cases} \qquad \text{and } X - \varphi(X) = \begin{cases} X \text{ if } X \leq d \\ 0 \text{ otherwise} \end{cases}$$

  - Term insurance with doubled capital in case of accidental death.
  - Endowment insurance.

Expected utility and mutual exclusivity

Theorem (Additivity property of mutual exclusive losses)

Consider a utlity function u, satisfying $u(w) = 0$. If X and Y are mutual exclusive losses, then

$$\mathbb{E}\left[u(w - X - Y)\right] = \mathbb{E}\left[u(w - X)\right] + \mathbb{E}\left[u(w - Y)\right].$$

- A general utility function u can always be standardized such that $u(w) = 0$.
- Interpretation:
  - The utility after bearing the loss $X + Y$ is the sum of the individual expected utilities.

Introduction: ordering of risks

- The perception of risk is captured in a utility function u:

$$u(x) = \text{moral value of having a wealth equal to } x.$$

- A decision maker is assumed to be an expected utility maximizer:
  - for a decision maker with utility function u, loss X is 'more preferable' than loss Y if:
  $$\mathbb{E}\left[u\left(w - X\right)\right] \geq \mathbb{E}\left[u\left(w - Y\right)\right].$$
  - there may exist another decision maker with utility function v, who prefers Y over X.
- The notion 'more preferable' depends on:
  - the distribution of the risk itself;
  - the risk preferences of a particular decision maker.

The concept 'more preferable'

- Equality in distribution:
  - Two r.v.'s X and Y are said to be equal in distribution if:

$$F_X(x) = F_Y(x), \text{ for all } x \in \mathbb{R}.$$

  - Notation: $X \overset{d}{=} Y$.
- If $X \overset{d}{=} Y$, **all** decision makers will be indifferent between risk X and risk Y.
- If $X \overset{d}{\neq} Y$, the notion 'more preferable' should be based on the distribution of the loss alone, not on a particular utility function.

Definition

- A decision maker's utility function u is in general unknown.
- Group all 'reasonable' decision makers in a class $\mathcal{U}$.
- $u(-X)$ represents the utility of a decision maker with zero initial wealth, after suffering the loss X.
- Integral stochastic order based on the class $\mathcal{U}$:

$$X \preceq_{\mathcal{U}} Y \Leftrightarrow \mathbb{E}\left[u(-X)\right] \geq \mathbb{E}\left[u(-Y)\right] \text{ for all } u \in \mathcal{U}.$$

- Interpretation:
  - All decision makers with *zero initial wealth* and belonging to the class $\mathcal{U}$ prefer the loss X over Y.

- Consider two losses X and Y, for which $X \preceq_{\mathcal{U}} Y$.
- Consider a decision maker with utility function u and *initial wealth w*.
  - The decision maker prefers X over Y if
  $$\mathbb{E}\left[u(w-X)\right] \geq \mathbb{E}\left[u(w-Y)\right]. \tag{1}$$
  - $X \preceq_{\mathcal{U}} Y$ does not necessarily imply (1).
- Assumption concerning $\mathcal{U}$:
  - Define the utility function v as: $v(x) = u(w+x)$.
  $$u \in \mathcal{U} \Rightarrow v \in \mathcal{U}.$$
- Interpretation:
  - The preference of X over Y does not depend on the initial wealth.

- Consider an insurer facing the risk X.
- The cdf of X will in general be unknown or too cumbersome to work with.
  - The only information available is that X belongs to some class:

$$X \in \mathcal{A}.$$

- Picking a particular member of $\mathcal{A}$ will lead to model risk.
- Making the wrong choice can lead to serious underestimation of the real risk.

Applications

- Replace the loss X by Y, such that for every $Z \in \mathcal{A}$[3]:

$$Z \preceq_{\mathcal{U}} Y.$$

- The r.v. Y describes a *worst case scenario*.
- Calculating actuarial quantities for Y is a *'prudent strategy'*.
- References:
    - Exotic option pricing: Schoutens, Simons & Tistaert (2004).
    - Risk measures: Barrieu & Scandolo (2013).

[3] For simplicity we assume that such Y exists.

Losses versus gains

- If X denotes a loss:
  - high positive values are big losses;
  - prefer loss X over Y if
  $$\mathbb{E}\left[u(w-X)\right] \geq \mathbb{E}\left[u(w-Y)\right].$$
  - $-X$ is a r.v. representing gains.
- If X denotes a gain:
  - negative values are losses;
  - prefer gain Y over X if
  $$\mathbb{E}\left[u(w+X)\right] \leq \mathbb{E}\left[u(w+Y)\right].$$
  - $-X$ is a loss r.v.

Definition (Stochastic dominance)

Two r.v.'s X and Y are ordered in the stochastic dominance sense, notation $X \preceq_{st} Y$ if

$$\mathbb{E}\left[u(-X)\right] \geq \mathbb{E}\left[u(-Y)\right],$$

for all non-decreasing function u.

- The class $\mathcal{U}$ is:

$$\mathcal{U} = \{u | u \text{ is a non-decreasing utility function}\}.$$

- $\mathcal{U}$ is the class of all decision makers who prefer more over less wealth.
- Interpretation:
  - If $X \preceq_{st} Y$, **all** decision makers will prefer X over Y.
  - Replacing loss X by loss Y is a prudent strategy.

Losses versus gains

- $u(x)$ is non-decreasing $\Leftrightarrow -u(-x)$ is non-decreasing.
  - For a non-decreasing utility function u, define the function v as
$$v(x) = -u(-x). \tag{2}$$
  - The function v is again a utility function in the class $\mathcal{U}$.
- Stochastic dominance in terms of gains:
$$X \preceq_{st} Y \Leftrightarrow \mathbb{E}\left[v(X)\right] \leq \mathbb{E}\left[v(Y)\right],$$
for v a non-decreasing utility function.
- Interpretation:
  - $X \preceq_{st} Y$ means that a gain Y is more attractive than a gain X.

Characterization in terms of the cdf

- Characterization of stochastic dominance:

$$X \preceq_{st} Y \Leftrightarrow F_X(x) \geq F_Y(x), \quad \text{for all } x \in \mathbb{R}.$$

- Other characterization:

$$X \preceq_{st} Y \Leftrightarrow \mathbb{P}\left[X > x\right] \leq \mathbb{P}\left[Y > x\right], \quad \text{for all } x \in \mathbb{R}.$$

- Interpretation:
  - For losses: prefer the risk which has the smallest upper tail and largest lower tail.
  - For gains: prefer the risk which has the largest upper tail and smallest lower tail.
- Smaller loss X is equivalent with a larger gain $-X$:

$$X \preceq_{st} Y \Leftrightarrow -Y \preceq_{st} -X.$$

Stochastic dominance and ordered means

- The expected value $\mathbb{E}[X]$:

$$\mathbb{E}[X] = -\int_{-\infty}^{0} F_X(x)\mathrm{d}x + \int_{0}^{+\infty} (1 - F_X(x))\,\mathrm{d}x.$$

- Difference in means in terms of cdf's:

$$\mathbb{E}[Y] - \mathbb{E}[X] = \int_{-\infty}^{+\infty} (F_X(x) - F_Y(x))\,\mathrm{d}x.$$

  - Exericse: Prove that this implication holds.

- Stochastic dominance implies ordered means:

$$X \preceq_{st} Y \Rightarrow \mathbb{E}[X] \leq \mathbb{E}[Y].$$

  - Exericse: Prove that this implication holds.

Capturing a distribution in a real number

Theorem

Consider two r.v.'s X and Y. Then the following statements are equivalent:

1. $$X \preceq_{st} Y \quad \text{and} \quad \mathbb{E}[X] = \mathbb{E}[Y].$$

2. $$X \stackrel{d}{=} Y$$

- Proof: Good exercise to try at home.
- Interpretation:
  - Consider two losses X and Y with $X \preceq_{st} Y$.
  - Then, the mean is sufficient to characterize the losses.
  - If $\mathbb{E}[X] = \mathbb{E}[Y]$, any decision maker will be indifferent between the losses X and Y.

Definition (Stop-loss order)

Two r.v.'s X and Y are ordered in the stop-loss order sense, notation $X \preceq_{sl} Y$ if

$$\mathbb{E}\left[u(-X)\right] \geq \mathbb{E}\left[u(-Y)\right],$$

for all non-decreasing and **concave** functions u.

- The class $\mathcal{U}$ is:

$$\mathcal{U} = \{u | u \text{ is a non-decreasing and } \textbf{concave} \text{ utility function}\}.$$

- $\mathcal{U}$ is the class of all **risk-averse** decision makers.
- Interpretation:
  - If $X \preceq_{sl} Y$, all risk-averse decision makers will prefer X over Y.
  - Replacing loss X by loss Y is a prudent strategy.

Losses versus gains

- Relating convex and concave functions:
  - The following statements are equivalent:
    - ★ $u(x)$ is non-decreasing and concave,
    - ★ $v(x) = -u(-x)$ is non-decreasing and convex.
- Alternative definition for stop-loss order:
  - $X \preceq_{sl} Y$ if, and only if,

$$\mathbb{E}[v(X)] \leq \mathbb{E}[v(Y)],$$

    for all non-decreasing convex functions v.
  - v is not a utility function of a **risk-averse** decision maker.
  - Stop-loss order has no interpretation in terms of gains when considering **risk-averse** decision makers.

Example: Reinsurance

- Reinsurance:
  - Total risk of an insurer $= X$.
  - The insurer moves the biggest losses to the reinsurer.
    - ⋆ Insurer pays the losses below K:

$$\text{Payments of Insurer} = \begin{cases} X, & \text{if } X \leq K \\ K, & \text{if } X > K \end{cases}.$$

    - ⋆ The reinsurer starts paying when the losses exceed the threshold K:

$$\begin{aligned} \text{Payments of Reinsurer} &= \begin{cases} 0, & \text{if } X \leq K \\ X - K, & \text{if } X > K \end{cases} \\ &\overset{\text{notation}}{=} (X-K)_+ \end{aligned}$$

- Expected payment of the **reinsurer**: $\mathbb{E}\left[(X-K)_+\right]$.
  - $\mathbb{E}\left[(X-K)_+\right]$ gives information about the big losses, which have to be paid by the reinsurer.

Example: Call option

- X denotes the price of a stock (e.g. Apple) at some future date T (e.g. one year).
- Call option
  - A **call option** with strike K and maturity T gives the buyer the right to buy the stock at time T for the price K.
  - The buyer will benefit from this product when the stock price increases.
- At maturity, the buyer will receive a pay-off equal to:

$$\text{Pay-off at maturity} = \begin{cases} 0, & \text{if } X \leq K \\ X - K, & \text{if } X > K \end{cases}$$
$$\overset{\text{notation}}{=} (X - K)_+ \,.$$

- The expected pay-off is given by:

$$\mathbb{E}\left[(X - K)_+\right].$$

Definition

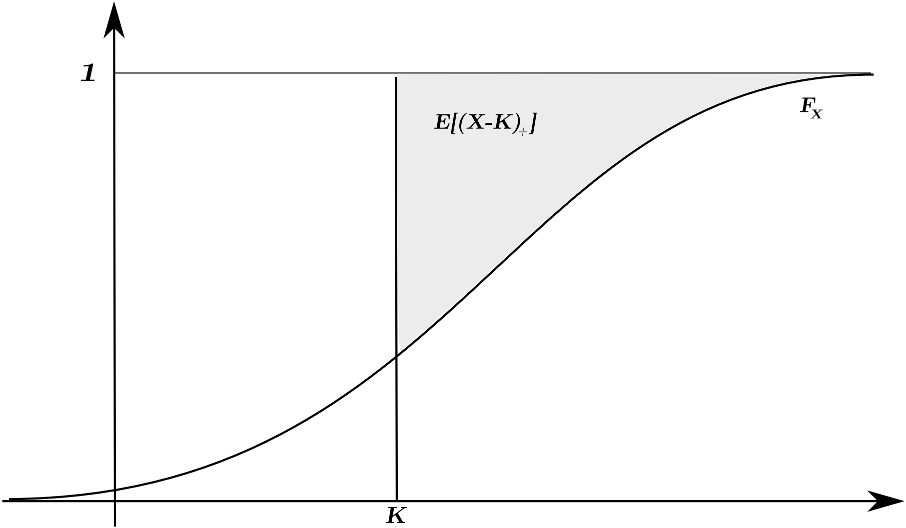
The stop-loss premium of the r.v. X with retention K is given by

$$\mathbb{E}\left[(X-K)_+\right].$$

- It can be proven that:

$$\mathbb{E}\left[(X-K)_+\right] = \int_K^{+\infty} \left(1 - F_X(x)\right) \mathrm{d}x.$$

- Interpretation:
  - Upper tail at level K.
  - $\mathbb{E}\left[(X-K)_+\right]$ is the surface between the cdf F_X and the constant function 1, from K to $+\infty$.


1
E[(X-K)+]
F_X
K

- Stop-loss transform:
  - $\pi_X(x) = \mathbb{E}\left[(X - x)_+\right].$
  - π_X is strictly decreasing and convex.
- The stop-loss transform characterizes the distribution of X:

$$\pi'_X(x+) = F_X(x) - 1,$$

  for $x \in \mathbb{R}$.
  - $\pi'_X(x+)$ is the right derivative of the function π_X in the point x.
- Alternative definition for stop-loss order:
  - $X \preceq_{sl} Y \Leftrightarrow \mathbb{E}\left[(X - K)_+\right] \leq \mathbb{E}\left[(Y - K)_+\right]$, for all $K \in \mathbb{R}$.
  - $X \preceq_{sl} Y$ means that X has uniformly smaller upper tails than Y.

Theorem (Crossing condition for stop-loss order)

If for two r.v.'s, there is a real number c such that

$$\begin{aligned} F_X(x) &\leq F_Y(x), \quad \text{for all } x < c, \\ F_X(x) &\geq F_Y(x), \quad \text{for all } x \geq c, \end{aligned}$$

and if also $\mathbb{E}[X] \leq \mathbb{E}[Y]$, then

$$X \preceq_{sl} Y.$$

- Exercise: Give a graphical proof of this Theorem.

Definition

Definition (Second degree stochastic dominance)

Two r.v.'s X and Y are ordered in the Second degree stochastic dominance sense, notation $X \preceq_{sst} Y$, if

$$\mathbb{E}\left[u(X)\right] \leq \mathbb{E}\left[u(Y)\right],$$

for all non-decreasing and **concave** functions u.

- The class $\mathcal{U}$ is:

$$\mathcal{U} = \{u | u \text{ is a non-decreasing and } \textbf{concave} \text{ utility function}\}.$$

- $\mathcal{U}$ is the class of all **risk-averse** decision makers.
- Interpretation:
  - If $X \preceq_{sl} Y$, all risk-averse decision makers will prefer gain Y over X.

Example: Put option

- X denotes the price of a stock (e.g. Apple) at some future date T (e.g. one year).
- Put Option
  - A **put option** with strike K and maturity T gives the buyer the right to sell the stock at time T for the price K.
  - The buyer will benefit from this product when the stock price decreases.
- At maturity, the buyer will receive a pay-off equal to:

$$\text{Pay-off at maturity} = \begin{cases} K - X, & \text{if } X \leq K \\ 0, & \text{if } X > K \end{cases}.$$

$$\overset{\text{notation}}{=} (K - X)_+$$

- The expected pay-off is given by:

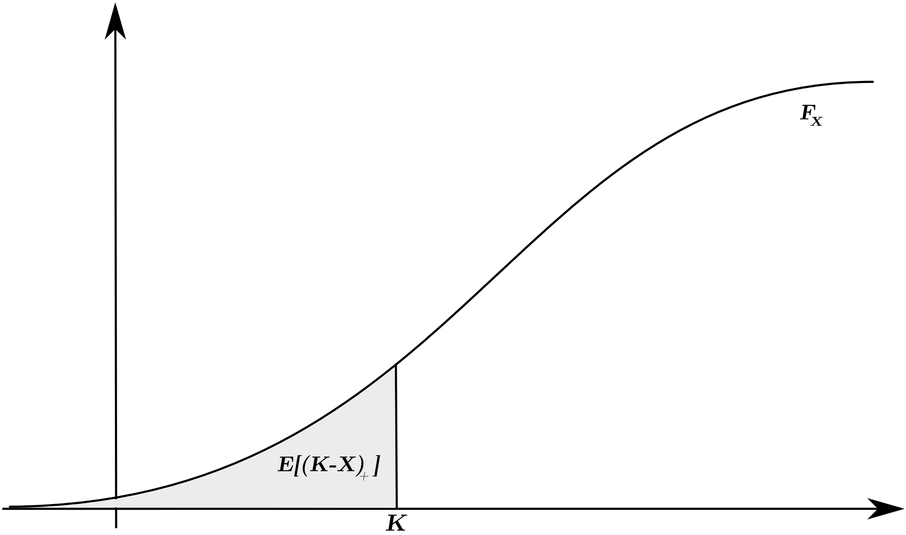
$$\mathbb{E}\left[(K - X)_+\right].$$

- Measure the lower tail using $\mathbb{E}\left[(K-X)_+\right]$.
- It can be proven that:

$$\mathbb{E}\left[(K-X)_+\right] = \int_{-\infty}^{K} F_X(x)\,\mathrm{d}x.$$

- Interpretation:
  - Lower tail at level K.
- lower tail transform:
  - $\lambda_X(x) = \mathbb{E}\left[(x-X)_+\right]$
  - A distribution is characterized by its lower tail transform:

$$\lambda_X'(x+) = F_X(x).$$



F_X
E[(K-X)_+]
K

Losses versus gains

- 2nd degree stochastic dominance has no interpretation in terms of losses for risk-averse decision makers.
- Stop-loss and 2nd degree stochastic dominance:

$$X \preceq_{sst} Y \Leftrightarrow -Y \preceq_{sl} -X$$

- Alternative definition:

$$X \preceq_{sst} Y \Leftrightarrow \mathbb{E}\left[(K-X)_+\right] \geq \mathbb{E}\left[(K-Y)_+\right], \quad \text{for all } K \in \mathbb{R}.$$

- Interpretation:
  - The larger the lower tails of a gain, the less attractive this gain has to be considered.

Definition (Crossing condition for sst-order)

If for two r.v.'s, there is a real number c such that

$$\begin{aligned} F_X(x) &\geq F_Y(x), \quad \text{for all } x < c, \\ F_X(x) &\leq F_Y(x), \quad \text{for all } x \geq c, \end{aligned}$$

and if also $\mathbb{E}[X] \leq \mathbb{E}[Y]$, then

$$X \preceq_{sst} Y.$$

- Exercise: Give a graphical proof of this Theorem.

Definition

Definition (Convex order)

Two r.v.'s X and Y are ordered in the convex order sense, notation $X \preceq_{cx} Y$ if

$$\mathbb{E}[X] = \mathbb{E}[Y] \quad \text{and} \quad \mathbb{E}[u(-X)] \geq \mathbb{E}[u(-Y)],$$

for all non-decreasing and **concave** functions u.

- $\mathcal{U}$ is the class of all risk-averse decision makers.
- Convex order gives the preferences of risk-averse decision makers between losses with the **same mean**.
- Interpretation:
  - A loss X is replaced by a less attractive r.v. Y, 'which is the same on average'.

- Connection between upper and lower tail:

$$\mathbb{E}\left[(X-K)_+\right] = \mathbb{E}\left[(K-X)_+\right] + \mathbb{E}\left[X\right] - K.$$

- This expression is called the **put-call parity** and has (in a modified form) wide applications in option pricing theory.
- The put-call parity can be proven in a graphical way, using the following expressions:

$$\begin{aligned}
\mathbb{E}\left[X\right] &= -\int_{-\infty}^{0} F_X(x)\mathrm{d}x + \int_{0}^{+\infty} \left(1 - F_X(x)\right)\mathrm{d}x, \\
\mathbb{E}\left[(X-K)_+\right] &= \int_{K}^{+\infty} \left(1 - F_X\left(x\right)\right)\mathrm{d}x, \\
\mathbb{E}\left[(K-X)_+\right] &= \int_{-\infty}^{K} F_X\left(x\right)\mathrm{d}x.
\end{aligned}$$

Ordered upper and lower tails

- Convex order and stop-loss order:

$$X \preceq_{cx} Y \Leftrightarrow \mathbb{E}[X] = \mathbb{E}[Y] \quad \text{and} \quad X \preceq_{sl} Y$$

- Convex order and upper tail transform:

$$X \preceq_{cx} Y \Leftrightarrow \begin{cases} \mathbb{E}[X] = \mathbb{E}[Y], \\ \mathbb{E}\left[(X-K)_+\right] \leq \mathbb{E}\left[(Y-K)_+\right], \quad \text{for all } K, \end{cases}$$

  - Follows directly from the definition of stop-loss order.

- Convex order and lower tail transform:

$$X \preceq_{cx} Y \Leftrightarrow \begin{cases} \mathbb{E}[X] = \mathbb{E}[Y], \\ \mathbb{E}\left[(K-X)_+\right] \leq \mathbb{E}\left[(K-Y)_+\right], \quad \text{for all } K, \end{cases}$$

  - Follows directly from the put-call parity.

Ordered upper and lower tails

- Convex order and second degree stochastic dominance:

$$X \preceq_{cx} Y \Leftrightarrow \mathbb{E}[X] = \mathbb{E}[Y] \quad \text{and} \quad Y \preceq_{sst} X$$

  - Follows directly from the definition of 2nd degree stochastic order.

- Convex order and lower and upper tail transforms:

$$X \preceq_{cx} Y \quad \Leftrightarrow \quad \begin{cases} X \preceq_{sl} Y, \\ Y \preceq_{sst} X, \end{cases} ,$$

$$\Leftrightarrow \quad \begin{cases} \mathbb{E}\left[(X-K)_+\right] \leq \mathbb{E}\left[(Y-K)_+\right], & \text{for all } K, \\ \mathbb{E}\left[(K-X)_+\right] \leq \mathbb{E}\left[(K-Y)_+\right], & \text{for all } K, \end{cases}$$

  - The proof of $\Rightarrow$ follows from previous relations.

Comparing variability of risks

- Convex order and convex/concave functions:

$$X \preceq_{cx} Y \Leftrightarrow \mathbb{E}\left[u(X)\right] \leq \mathbb{E}\left[u(Y)\right],$$

  for all convex functions u.

- $X \preceq_{cx} Y$ implies
  - Y has heavier upper tails than X,
  - Y has heavier lower tails than X.
- Convex order has an interpretation in terms of losses and gains.

$$X \preceq_{cx} Y \Leftrightarrow -X \preceq_{cx} -Y.$$

- R.v. X is 'less variable' than r.v. Y.

Theorem

If for two r.v.'s X and Y a real number c exists such that

$$\begin{aligned} F_X(x) &\leq F_Y(x) \text{ for all } x < c, \\ F_X(x) &\geq F_Y(x) \text{ for } x \geq c, \end{aligned} \tag{3}$$

and moreover $\mathbb{E}[X] = \mathbb{E}[Y]$, then $X \preceq_{cx} Y$.

- Exercise:

Lemma

For any $a \in \mathbb{R}$, $u(X)$ can be expressed as

$$u(X) = u(a) + u'(a)(X - a) + \int_{-\infty}^{a} u''(K)(K - X)_{+}\, dK + \int_{a}^{+\infty} u''(K)(X - K)_{+}\, dK. \quad (4)$$

- At a certain time, one has to pay the amount $u(X)$.
- The pay-off $u(X)$ can be decomposed using:
  - pay-off of call options: $(X - K)_{+}$,
  - pay-off of put options: $(K - X)_{+}$.
- Pay-offs of the form $(X - K)_{+}$ and $(K - X)_{+}$ are the building blocks for more complex pay-offs.

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[4] see Carr & Madan (2001) and Cheung, Dhaene, Kukush & Linders (2013)

Lemma

For a r.v. X, we can decompose $\mathbb{E}\left[u\left(X\right)\right]$ as

$$\mathbb{E}\left[u\left(X\right)\right] = u\left(\mathbb{E}\left[X\right]\right) + \int_{-\infty}^{\mathbb{E}[X]} u''\left(K\right)\mathbb{E}\left[\left(K - X\right)_+\right] dK + \int_{\mathbb{E}[X]}^{+\infty} u''\left(K\right)\mathbb{E}\left[\left(X - K\right)_+\right] dK.$$

- For a risk-averse decision maker, the expected utility $\mathbb{E}\left[u\left(X\right)\right]$ can be written as

  the utility of $\mathbb{E}[X]$ + portion of the tails.

- If $u''(x) \leq 0$, we immediately find Jensen's inequality:

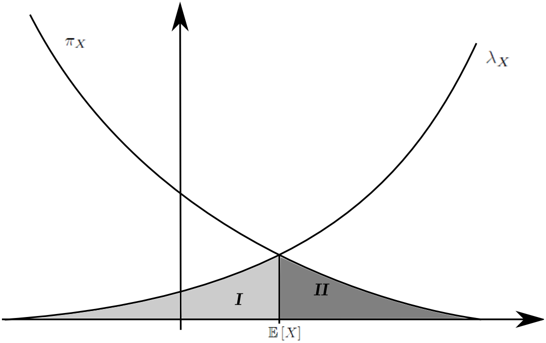
$$\mathbb{E}\left[u\left(X\right)\right] \leq u\left(\mathbb{E}\left[X\right]\right).$$

Example: Variance

- Variance decomposition formula
  - Take $u(x) = (x - \mathbb{E}[X])^2$, then:

$$\frac{1}{2}\mathrm{Var}[X] = \int_{-\infty}^{\mathbb{E}[X]} \mathbb{E}\left[(K-X)_+\right] \mathrm{d}K + \int_{\mathbb{E}[X]}^{+\infty} \mathbb{E}\left[(X-K)_+\right] \mathrm{d}K.$$

- Graphical representation:



Capturing a distribution in a real number

- Convex ordered r.v.'s:
  - We can prove that $X \preceq_{cx} Y$, then

$$\int_{-\infty}^{\infty} \left(\mathbb{E}\left[(Y-K)_+\right] - \mathbb{E}\left[(X-K)_+\right]\right) \mathrm{d}K = \frac{1}{2}\left(\mathrm{Var}\left[Y\right] - \mathrm{Var}\left[X\right]\right).$$

  - Comparing variances is meaningful when comparing SL-premiums of convex ordered r.v.'s.
  - If $X \preceq_{cx} Y$, then Var$[X] \leq$Var$[Y]$.
- The following statements are equivalent:
  - $X \preceq_{cx} Y$ and Var$[Y] =$ Var$[X]$
  - $X \overset{\mathrm{d}}{=} Y$.

Theorem

Consider the r.v.'s X and Y. Let u be a strictly concave function such that $\mathbb{E}\left[u\left(Y\right)\right]$ is finite.
Then

$$X \preceq_{cx} Y \text{ and } \mathbb{E}\left[u\left(X\right)\right] = \mathbb{E}\left[u\left(Y\right)\right] \quad (5)$$

is equivalent with

$$X \overset{\mathrm{d}}{=} Y. \quad (6)$$

- The function u has to have an absolutely continuous derivative to ensure all integrals are well-defined.
- The results hold for concave/convex twice differentiable utility functions.
- The condition $\mathbb{E}\left[u\left(X\right)\right] = \mathbb{E}\left[u\left(Y\right)\right]$ can be replaced by $\mathbb{E}\left[u\left(-X\right)\right] = \mathbb{E}\left[u\left(-Y\right)\right]$.

- Consider a decision maker with a strictly concave and twice differentiable utility function:

$$u''(x) < 0.$$

- He has to choose between two convex ordered gains:

$$X \preceq_{cx} Y.$$

- The decision maker values X using the **whole** distribution.
- If he prefers X over Y:

$$\mathbb{E}\left[u\left(X\right)\right] > \mathbb{E}\left[u\left(Y\right)\right],$$

  then X and Y cannot be equal (in distribution).
- If he is indifferent between X and Y, any other decision maker will be indifferent.

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