



Foundations of Quantitative Risk Measurement

Chapter 1: Expected Utility Theory¹

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1. Introduction
2. Expected Utility Theory
3. Expected utility and premium calculation
4. Expected utility and mutual exclusivity

- Examples of decision making problems:
 - ▶ *Individual*: bear a risk or insure it (partially)?
 - ▶ *Insurer*: accept a risk or not?
 - ▶ *Insurer*: reinsure (part of) the accepted risks?
- Optimal choice of the decision maker depends on:
 - ▶ his initial wealth,
 - ▶ his risk appetite.

- Theories of choice under risk:
 - ▶ Expected utility theory: Cramer (1728), Bernouilli 1738), Von Neumann & Morgenstern (1947).
 - ▶ Dual theory of choice under risk: Yaari (1987), Roëll (1987), Schmeidler (1989).
- Common properties of these theories:
 - ▶ Preference relations of a decision maker are qualitative in nature,
 - ▶ but follow from comparing numerical quantities.

Conventions and definitions

- f is a real-valued function.

$$f : I \rightarrow \mathbb{R},$$

- ▶ I is an interval in \mathbb{R} .

- ★ $I = [a, b], [a, +\infty), (-\infty, b]$ or $(-\infty, +\infty)$.

- ▶ f is a real-valued function.

- Non-decreasing functions:

- ▶ The function f is non-decreasing if the following holds

$$x \leq y \rightarrow f(x) \leq f(y), \text{ for } x, y \in \mathbb{R}.$$

- ▶ f is non-increasing if $-f$ is non-decreasing.

Conventions and definitions

- Continuous functions:

- ▶ The function f is continuous on $[a, b]$ if it is
 - ★ continuous on (a, b) ;
 - ★ right continuous at a ;
 - ★ left continuous at b .

- Differentiable functions:

- ▶ The function f is differentiable on $[a, b]$ if it is
 - ★ differentiable on (a, b) ;
 - ★ differentiable from the right continuous at a ;
 - ★ differentiable from the left at b .

Random variables and cumulative distribution functions

- Probability space: $(\Omega, \mathcal{F}, \mathbb{P})$
 - ▶ Ω : all future states of nature.
 - ★ $\omega \in \Omega$: a possible realization of a random experiment.
 - ▶ \mathcal{F} : set of events.
 - ★ Collection of subsets of Ω .
 - ▶ \mathbb{P} : probability measure assigning the likelihood to each event.
 - ★ Attach a probability to each event.
 - ★ $\mathbb{P} : \mathcal{F} \rightarrow [0, 1]$.
 - ★ $\mathbb{P}[\Omega] = 1$.

Random variables and cumulative distribution functions

- A random variable X defined on $(\Omega, \mathcal{F}, \mathbb{P})$ is a *function* which attaches a *real number* to each possible outcome:

$$X : \Omega \longrightarrow \mathbb{R}.$$

- ▶ ω describes the state of a random phenomenon.
- ▶ $X[\omega]$ is a single aspect of the state ω .
- Define the set $X^{-1}(I)$ as:

$$X^{-1}(I) = \{\omega \in \Omega \mid X[\omega] \in I\}, \text{ where } I \subseteq \mathbb{R}.$$

- ▶ Assumption: The random variable X is a **measurable function** on the measurable space (Ω, \mathcal{F}) . Hence: $X^{-1}(I) \in \mathcal{F}$.

Random variables and cumulative distribution functions

- What is the probability that X lies in the interval B ?
 - ▶ Probability function \mathbb{P} assigns probabilities to subsets of Ω .
 - ▶ $\mathbb{P} [X^{-1}(B)] =$ probability that X takes a value in B .
 - ▶ Notation:

$$\mathbb{P} [X \in B] = \mathbb{P} [X^{-1}(B)] .$$

- We assume that the probability $\mathbb{P}[X \in B]$ is known.
- *The only uncertainty when considering a future random loss is the uncertainty about its particular future outcome, not the uncertainty about its 'distribution'.*

Random variables and cumulative distribution functions

- Cumulative distribution function (cdf) F_X of the r.v. X :

$$F_X(x) = \mathbb{P}[X \leq x], \text{ for } x \in \mathbb{R}.$$

- ▶ F_X is non-decreasing and right continuous.
- Assume F_X has a jump of size $\Delta(x)$ in x :
 - ▶ $\Delta(x) = F_X(x) - F_X(x-)$.
 - ▶ $\Delta(x)$ is zero if F_X is continuous in x .
 - ▶ For all $x \in \mathbb{R}$:

$$\mathbb{P}[X = x] = \Delta(x).$$

Random variables and cumulative distribution functions

- The expectation:

$$\mathbb{E} [g(X)] = \int_I g(x) dF_X(x).$$

▶ $X : \Omega \rightarrow I$ and $g : I \rightarrow \mathbb{R}$.

- Assumption:

▶ The expectation of X is finite:

$$\mathbb{E}[X] < \infty.$$

▶ Expressions involving $\mathbb{E} [g(X)]$ are provided this quantity is well-defined and finite.

Random variables and cumulative distribution functions

- If F_X has only a discrete part:

$$\mathbb{E} [g(X)] = \sum_y g(y) \Delta (y) = \sum_y g(y) \mathbb{P} [X = y].$$

- If F_X has a discrete and continuous part:

$$\mathbb{E} [g(X)] = \int_I g(x) f_X (x) dx + \sum_y g(y) \Delta (y).$$

- ▶ $f_X (x) dx =$ probability that X takes a value in the $[x, x + dx]$.

The St. Petersburg Paradox

- Problem:

- ▶ *A fair coin is tossed repeatedly until it lands head up. The income you receive is equal to 2^n if the first head appears on the n -th toss. How much are you willing to pay for this game?*

- Expected gain:

- ▶ Assume that the coin is fair.
- ▶ Probability to win the amount 2^n is $\frac{1}{2^n}$.
- ▶ The expected gain:

$$\sum_{n=1}^{+\infty} (2^n) \times \frac{1}{2^n} = \sum_{n=1}^{+\infty} 1 = +\infty.$$

The St. Petersburg Paradox

- The expected income from this game is $+\infty$, however ...
 - ▶ A decision maker will not pay $+\infty$.
 - ▶ The price to play this game will be modest.
- Conclusion:
 - ▶ The expectation is not (always) a good method to rank uncertain outcomes.

The St. Petersburg Paradox

- Classical expected utility theory:
 - ▶ Each decision maker assigns a utility $u(x)$ to any fortune of amount x .
 - ▶ Utility functions are of a subjective nature.
 - ▶ 'Reasonable' utility functions share common properties:
 - ★ non-decreasingness,
 - ★ decreasing marginal utility.
- Expected utility and insurance:
 - ▶ Why is an individual willing to pay a premium larger than the average expected loss?
 - ▶ Why are certain insurance covers to be preferred over others?
 - ▶ Behavior of insureds:
 - ★ moral hazard,
 - ★ anti-selection.

The St. Petersburg Paradox

- Consider a decision maker with initial fortune w .
- He attaches a utility $u(x)$ to a fortune x .
- The price to play the game is P .
- Assume our agent wins after n throws.
 - ▶ His utility if he wins after n throws: $u(w - P + 2^n)$.
 - ▶ Probability to win after n throws: $\frac{1}{2^n}$.
- Expected utility:
 - ▶ At initiation, the utility he will reach if he plays the game is unknown.
 - ▶ Expected utility:

$$\sum_{n=1}^{+\infty} u(w - P + 2^n) \frac{1}{2^n}.$$

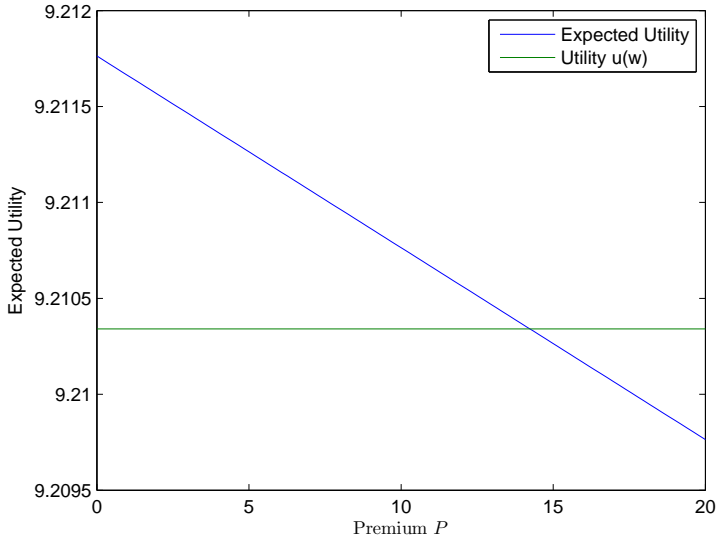
The St. Petersburg Paradox

- The decision maker is an *expected utility maximizer*.
- If he doesn't play the game, his utility is $u(w)$.
- He is willing to play the coin tossing game for a price P if

$$u(w) \leq \sum_{n=1}^{+\infty} u(w - P + 2^n) \frac{1}{2^n}$$

- ▶ G. Cramer: $u(x) = \sqrt{x}$.
- ▶ D. Bernoulli: $u(x) = \ln x$.
- Example:
 - ▶ Take $w = 10000$ and $u(x) = \ln x$.
 - ▶ Then $P = 14.2385$. (Check this using MatLab or Excel!)

St. Petersburg Paradox



Transformed wealth levels

Definition (Utility function)

A **utility function** u is a real-valued *non-decreasing* function asserting a decision maker's utility-of-wealth $u(x)$ to each possible level of wealth x .

- Decision makers have *non-negative marginal utility*: more wealth is always preferred over less wealth.
- In general, different decision makers will have different utility functions.
- We study classes of decision makers, which all share some common risk preferences

Transformed wealth levels

- Consider a decision maker having initial wealth w and facing a loss X .
- Wealth after suffering the loss X :

$$w - X.$$

- Utility level after suffering the loss X

$$u(w - X).$$

▶ $u(w - X)$ is a r.v.

- The expected utility is the quantity:

$$\mathbb{E} [u(w - X)].$$

Transformed wealth levels

- The expected utility hypothesis:

Prefer loss X over loss $Y \iff \mathbb{E}[u(w - X)] \geq \mathbb{E}[u(w - Y)]$,

Indifferent between X and $Y \iff \mathbb{E}[u(w - X)] = \mathbb{E}[u(w - Y)]$.

- ▶ Relations as above hold 'provided the expectations exist'.
 - ▶ The decision maker is said to be an *expected utility maximizer*.
 - ▶ Indifferent between losses with the same distribution.
- Standardized utility functions:
 - ▶ A utility function only needs to be determined up to positive linear transformations.
 - ★ Exercise: prove this statement!
 - ▶ Standardize the utility function u :

$$u(x_0) = 0 \text{ and } u'(x_0) = 1, \text{ for some } x_0 \in \mathbb{R}.$$

Transformed wealth levels

- Axiomatic framework - Von Neumann & Morgenstern (1947):
 - ▶ *Any decision maker whose behavior is in accordance with a given set of 'rational' axioms, is an expected utility maximizer.*
- The 'independence axiom':
 - ▶ For any random losses X , Y and Z and for any Bernoulli r.v. I , independent of X , Y and Z , one has:
$$\text{Prefer loss } X \text{ over loss } Y$$
$$\Rightarrow \text{Prefer loss } IX + (1 - I)Z \text{ over loss } IY + (1 - I)Z$$
 - ▶ Example.

Definition (concave function)

A real-valued function f , defined on the interval $I \subseteq \mathbb{R}$, is **concave** on I if for any $x_1, x_2 \in I$ and any $t \in [0, 1]$,

$$f(tx_1 + (1-t)x_2) \geq tf(x_1) + (1-t)f(x_2)$$

- f is **convex** on the interval I if $(-f)$ is concave on I .
- Assume f is twice differentiable:
 - ▶ f is concave $\Leftrightarrow f''(x) \leq 0$, for all $x \in I$.
 - ▶ f is convex $\Leftrightarrow f''(x) \geq 0$, for all $x \in I$.
- f is concave $\Rightarrow f$ is continuous.

2 – Expected utility theory

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Expected utility and risk aversion

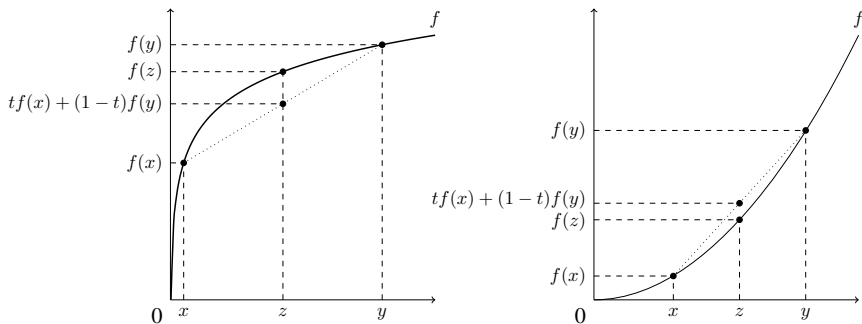


Figure: A concave (left plot) and a convex (right plot) function f where we have taken $z = tx + (1 - t)y$.

Definition (Risk averse decision makers)

A decision maker is **risk averse** if his utility function u is concave on its domain.

- Risk averse decision makers have *decreasing marginal utility*.
 - ▶ Assume you gain the amount Δ .
 - ▶ Increase in utility: $u(x + \Delta) - u(x)$.
 - ▶ For risk averse decision makers, the increase in utility is a decreasing function of the wealth level x .
- Interpretation:
 - ▶ As more wealth is available, less 'moral value' is placed on earning an additional Euro.

Expected utility and risk aversion

- A risk is an event solely due to the whims of fate that may or may not take place
 - ▶ and that brings about some financial loss,
 - ▶ or a financial gain.
- Examples:
 - ▶ For an insurer, a risk is a potential loss (e.g. car insurance);
 - ▶ For an investor, a risk is a potential gain (e.g. investment).
- A risk always contains uncertainty:
 - ▶ The event that may or may not take place,
 - ▶ or the severity of the consequences of its occurrence,
 - ▶ or the moment of its occurrence.
- Risk vs. loss:
 - ▶ 'Risk' and 'loss' are synonyms.

Theorem (Jensen's inequality (1906))

$$f \text{ is concave} \Rightarrow \mathbb{E} [f(Y)] \leq f(\mathbb{E}[Y])$$

- Corollary: If u is a concave utility function, then

$$\mathbb{E} [u(w - X)] \leq u(w - \mathbb{E}[X]).$$

- ▶ Exercise: prove this inequality.
- The risk averse decision maker's attitude towards risk:
 - ▶ Prefer certainty over uncertainty with the same expectation.
- The risk averse decision maker's attitude towards wealth:
 - ▶ Decreasing marginal utility.

Expected utility and risk aversion

- Definition:

A decision maker is *risk neutral* if

$$u(x) = ax + b$$

for given constants $a > 0$ and b .

- In this case, the expected utility hypothesis coincides with comparing expected values.
- The Arrow-Pratt measure of absolute risk aversion:

$$r(x) = \frac{-u''(x)}{u'(x)} = -\frac{d}{dx} \ln(u'(x))$$

For any risk averse decision maker, we have that $r \geq 0$.

- Risk averse individual:
 - ▶ facing a loss $X \geq 0$,
 - ▶ utility function $u(x)$,
 - ▶ initial wealth w .
- Risk averse insurer:
 - ▶ accepts X for a premium P ,
 - ▶ utility function U ,
 - ▶ initial wealth W .
- Under what conditions is an insurance contract feasible?
 - ▶ From the viewpoint of the individual,
 - ▶ from the viewpoint of the insurer.

- Viewpoint of the individual:

- ▶ He is only willing to underwrite the insurance if

$$u(w - P) \geq \mathbb{E}[u(w - X)].$$

- ▶ There exists always a premium P^M such that

$$u(w - P^M) = \mathbb{E}[u(w - X)].$$

- ★ u is non-decreasing.
- ★ u is concave, hence also continuous.
- ★ P^M is the maximum premium the insured is willing to pay.

- From Jensen's inequality:

$$P^M \geq \mathbb{E}[X].$$

- ▶ Exercise: prove this inequality.

- Viewpoint of the insurer:

- ▶ He is willing to insure the risk X at a premium P if

$$U(W) \leq \mathbb{E}[U(W + P - X)].$$

- ▶ Minimal premium P^m he requires follows from

$$U(W) = \mathbb{E}[U(W + P^m - X)].$$

- ▶ From Jensen's inequality:

$$P^m \geq \mathbb{E}[X].$$

★ Exercise: prove this inequality.

- Condition for an insurance contract to be feasible:

$$P^m \leq P \leq P^M$$

- Definition:

- ▶ The random vector (X_1, X_2, \dots, X_n) is said to be mutually exclusive if the following conditions hold:

$$\mathbb{P} [X_i \neq 0, X_j \neq 0] = 0, \quad \forall i \neq j$$

- Examples of mutual exclusive couples:

- ▶ Insurance with a franchise deductible:

$$\varphi(X) = \begin{cases} 0 & \text{if } X \leq d \\ X & \text{otherwise} \end{cases} \quad \text{and } X - \varphi(X) = \begin{cases} X & \text{if } X \leq d \\ 0 & \text{otherwise} \end{cases}$$

- ▶ Term insurance with doubled capital in case of accidental death.
- ▶ Endowment insurance.

Theorem (Additivity property of mutual exclusive losses)

Consider a utility function u , satisfying $u(w) = 0$. If X and Y are mutual exclusive losses, then

$$\mathbb{E} [u(w - X - Y)] = \mathbb{E} [u(w - X)] + \mathbb{E} [u(w - Y)].$$

- A general utility function u can always be standardized such that $u(w) = 0$.
- Interpretation:
 - ▶ The utility after bearing the loss $X + Y$ is the sum of the individual expected utilities.