KU LEUVEN

Foundations of Quantitative Risk Measurement

Chapter 2: Integral Stochastic Orders[1]

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[1] Chapter 2 from 'Managing and measuring actuarial risks', Dhaene, J., Denuit, M., Goovaerts, M., Kaas, R. & Linders, D. (2017), To be published.

1. Introduction
2. Stochastic dominance
3. Stop-loss order
4. Second degree stochastic dominance
5. Convex order
6. Convex order and equality in distribution

- The perception of risk is captured in a utility function u:

$$u(x) = \text{moral value of having a wealth equal to } x.$$

- A decision maker is assumed to be an expected utility maximizer:
  - for a decision maker with utility function u, loss X is 'more preferable' than loss Y if:
  $$\mathbb{E}\left[u\left(w - X\right)\right] \geq \mathbb{E}\left[u\left(w - Y\right)\right].$$
  - there may exist another decision maker with utility function v, who prefers Y over X.
- The notion 'more preferable' depends on:
  - the distribution of the risk itself;
  - the risk preferences of a particular decision maker.

Classifying risks

- Equality in distribution:
  - Two r.v.'s X and Y are said to be equal in distribution if:

$$F_X(x) = F_Y(x), \text{ for all } x \in \mathbb{R}.$$

  - Notation: $X \stackrel{\text{d}}{=} Y$.
- If $X \stackrel{\text{d}}{=} Y$, **all** decision makers will be indifferent between risk X and risk Y.
- If $X \stackrel{\text{d}}{\neq} Y$, the notion 'more preferable' should be based on the distribution of the loss alone, not on a particular utility function.

Definition

- A decision maker's utility function u is in general unknown.
- Group all 'reasonable' decision makers in a class $\mathcal{U}$.
- $u(-X)$ represents the utility of a decision maker with zero initial wealth, after suffering the loss X.
- Integral stochastic order based on the class $\mathcal{U}$:

$$X \preceq_{\mathcal{U}} Y \Leftrightarrow \mathbb{E}\left[u(-X)\right] \geq \mathbb{E}\left[u(-Y)\right] \text{ for all } u \in \mathcal{U}.$$

- Interpretation:
  - All decision makers with *zero initial wealth* and belonging to the class $\mathcal{U}$ prefer the loss X over Y.

- Consider two losses X and Y, for which $X \preceq_{\mathcal{U}} Y$.
- Consider a decision maker with utility function u and *initial wealth w*.
  - The decision maker prefers X over Y if

$$\mathbb{E}\left[u(w-X)\right] \geq \mathbb{E}\left[u(w-Y)\right]. \quad (1)$$

  - $X \preceq_{\mathcal{U}} Y$ does not necessarily imply (1).
- Assumption concerning $\mathcal{U}$:
  - Define the utility function v as: $v(x) = u(w+x)$.

$$u \in \mathcal{U} \Rightarrow v \in \mathcal{U}.$$

- Interpretation:
  - The preference of X over Y does not depend on the initial wealth.

Application: worst-case scenarios

- Consider an insurer facing the risk X.
- The cdf of X will in general be unknown or too cumbersome to work with.
  - The only information available is that X belongs to some class:

$$X \in \mathcal{A}.$$

- Picking a particular member of $\mathcal{A}$ will lead to model risk.
- Making the wrong choice can lead to serious underestimation of the real risk.

Application: worst-case scenarios

- Replace the loss X by Y, such that for every $Z \in \mathcal{A}$[2]:

$$Z \preceq_{\mathcal{U}} Y.$$

- The r.v. Y describes a *worst case scenario*.
- Calculating actuarial quantities for Y is a *'prudent strategy'*.
- References:
  - Exotic option pricing: Schoutens, Simons & Tistaert (2004).
  - Risk measures: Barrieu & Scandolo (2013).

[2] For simplicity we assume that such Y exists.

Losses versus gains

- If X denotes a loss:
  - high positive values are big losses;
  - prefer loss X over Y if
$$\mathbb{E}\left[u(w-X)\right] \geq \mathbb{E}\left[u(w-Y)\right].$$
  - $-X$ is a r.v. representing gains.
- If X denotes a gain:
  - negative values are losses;
  - prefer gain Y over X if
$$\mathbb{E}\left[u(w+X)\right] \leq \mathbb{E}\left[u(w+Y)\right].$$
  - $-X$ is a loss r.v.

Definition (Stochastic dominance)

Two r.v.'s X and Y are ordered in the stochastic dominance sense, notation $X \preceq_{st} Y$ if

$$\mathbb{E}\left[u(-X)\right] \geq \mathbb{E}\left[u(-Y)\right],$$

for all non-decreasing function u.

- The class $\mathcal{U}$ is:

$$\mathcal{U} = \{u | u \text{ is a non-decreasing utility function}\}.$$

- $\mathcal{U}$ is the class of all decision makers who prefer more over less wealth.
- Interpretation:
  - If $X \preceq_{st} Y$, **all** decision makers will prefer X over Y.
  - Replacing loss X by loss Y is a prudent strategy.

Losses versus gains

- $u(x)$ is non-decreasing $\Leftrightarrow -u(-x)$ is non-decreasing.
  - For a non-decreasing utility function u, define the function v as

$$v(x) = -u(-x). \tag{2}$$

  - The function v is again a utility function in the class $\mathcal{U}$.
- Stochastic dominance in terms of gains:

$$X \preceq_{st} Y \Leftrightarrow \mathbb{E}\left[v(X)\right] \leq \mathbb{E}\left[v(Y)\right],$$

  for v a non-decreasing utility function.
- Interpretation:
  - $X \preceq_{st} Y$ means that a gain Y is more attractive than a gain X.

Characterization in terms of the cdf

- Characterization of stochastic dominance:

$$X \preceq_{st} Y \Leftrightarrow F_X(x) \geq F_Y(x), \quad \text{for all } x \in \mathbb{R}.$$

- Other characterization:

$$X \preceq_{st} Y \Leftrightarrow \mathbb{P}\left[X > x\right] \leq \mathbb{P}\left[Y > x\right], \quad \text{for all } x \in \mathbb{R}.$$

- Interpretation:
  - For losses: prefer the risk which has the smallest upper tail and largest lower tail.
  - For gains: prefer the risk which has the largest upper tail and smallest lower tail.
- Smaller loss X is equivalent with a larger gain $-X$:

$$X \preceq_{st} Y \Leftrightarrow -Y \preceq_{st} -X.$$

Stochastic dominance and ordered means

- The expected value $\mathbb{E}[X]$:

$$\mathbb{E}[X] = -\int_{-\infty}^{0} F_X(x)\mathrm{d}x + \int_{0}^{+\infty} \left(1 - F_X(x)\right)\mathrm{d}x.$$

- Difference in means in terms of cdf's:

$$\mathbb{E}[Y] - \mathbb{E}[X] = \int_{-\infty}^{+\infty} \left(F_X(x) - F_Y(x)\right)\mathrm{d}x.$$

  - Exericse: Prove this expression.

- Stochastic dominance implies ordered means:

$$X \preceq_{st} Y \Rightarrow \mathbb{E}[X] \leq \mathbb{E}[Y].$$

  - Exericse: Prove that this implication holds.

Definition (Stop-loss order)

Two r.v.'s X and Y are ordered in the stop-loss order sense, notation $X \preceq_{sl} Y$ if

$$\mathbb{E}\left[u(-X)\right] \geq \mathbb{E}\left[u(-Y)\right],$$

for all non-decreasing and **concave** functions u.

- The class $\mathcal{U}$ is:

$$\mathcal{U} = \{u | u \text{ is a non-decreasing and } \textbf{concave} \text{ utility function}\}.$$

- $\mathcal{U}$ is the class of all **risk-averse** decision makers.
- Interpretation:
  - If $X \preceq_{sl} Y$, all risk-averse decision makers will prefer X over Y.
  - Replacing loss X by loss Y is a prudent strategy.

Losses versus gains

- Relating convex and concave functions:
  - The following statements are equivalent:
    - $u(x)$ is non-decreasing and concave,
    - $v(x) = -u(-x)$ is non-decreasing and convex.
- Alternative definition for stop-loss order:
  - $X \preceq_{sl} Y$ if, and only if,

$$\mathbb{E}\left[v(X)\right] \leq \mathbb{E}\left[v(Y)\right],$$

    for all non-decreasing convex functions v.
  - v is not a utility function of a **risk-averse** decision maker.
  - Stop-loss order has no interpretation in terms of gains when considering **risk-averse** decision makers.

Example: Reinsurance

- Reinsurance:
  - Total risk of an insurer $= X$.
  - The insurer moves the biggest losses to the reinsurer.
    - ⋆ Insurer pays the losses below K:

$$\text{Payments of Insurer} = \begin{cases} X, & \text{if } X \leq K \\ K, & \text{if } X > K \end{cases}.$$

    - ⋆ The reinsurer starts paying when the losses exceed the threshold K:

$$\text{Payments of Reinsurer} = \begin{cases} 0, & \text{if } X \leq K \\ X - K, & \text{if } X > K \end{cases} \overset{\text{notation}}{=} (X-K)_+$$

- Expected payment of the **reinsurer**: $\mathbb{E}\left[(X-K)_+\right]$.
  - $\mathbb{E}\left[(X-K)_+\right]$ gives information about the big losses, which have to be paid by the reinsurer.

Example: Call option

- X denotes the price of a stock (e.g. Apple) at some future date T (e.g. one year).
- Call option
  - A **call option** with strike K and maturity T gives the buyer the right to buy the stock at time T for the price K.
  - The buyer will benefit from this product when the stock price increases.
- At maturity, the buyer will receive a pay-off equal to:

$$\text{Pay-off at maturity} = \begin{cases} 0, & \text{if } X \leq K \\ X - K, & \text{if } X > K \end{cases} \overset{\text{notation}}{=} (X - K)_+ .$$

- The expected pay-off is given by:

$$\mathbb{E}\left[(X - K)_+\right].$$

Definition of stop-loss premium

Definition

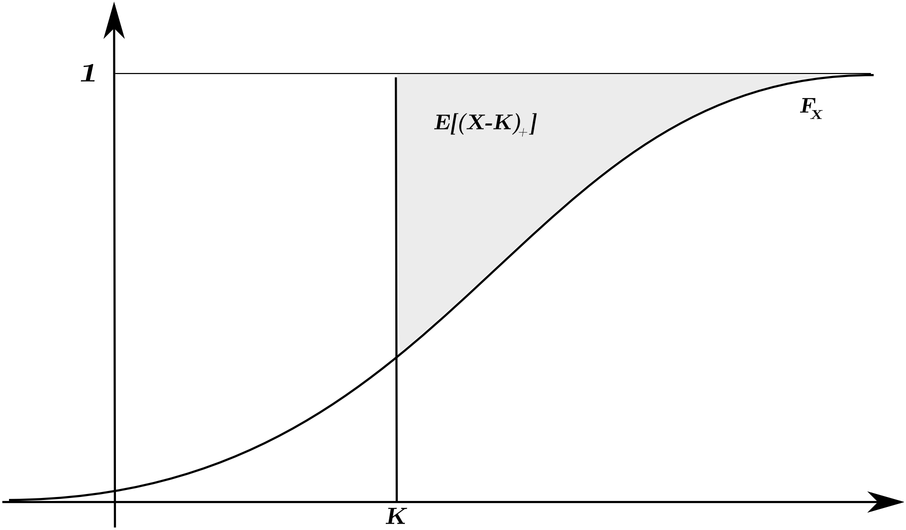
The stop-loss premium of the r.v. X with retention K is given by

$$\mathbb{E}\left[(X-K)_+\right].$$

- It can be proven that:

$$\mathbb{E}\left[(X-K)_+\right] = \int_K^{+\infty} (1 - F_X(x))\, \mathrm{d}x.$$

- Interpretation:
  - Upper tail at level K.
  - $\mathbb{E}\left[(X-K)_+\right]$ is the surface between the cdf F_X and the constant function 1, from K to $+\infty$.


1
E[(X-K)+]
F_X
K

- Stop-loss transform:
  - $\pi_X(x) = \mathbb{E}\left[(X - x)_+\right]$.
  - π_X is strictly decreasing and convex.
- The stop-loss transform characterizes the distribution of X:

$$\pi'_X(x+) = F_X(x) - 1,$$

for $x \in \mathbb{R}$.
  - $\pi'_X(x+)$ is the right derivative of the function π_X in the point x.
- Stop-loss premiums and distributions:

$$\pi_X(K) = \pi_Y(K) \text{ for all } K \in \mathbb{R} \Leftrightarrow X \overset{d}{=} Y.$$

- Alternative definition for stop-loss order:

$$X \preceq_{sl} Y \Leftrightarrow \mathbb{E}\left[(X-K)_+\right] \leq \mathbb{E}\left[(Y-K)_+\right], \text{ for all } K \in \mathbb{R}.$$

- Interpretation:
  - $X \preceq_{sl} Y$ means that X has uniformly smaller **upper** tails than Y.
  - $X \preceq_{sl} Y$ does not say anything about the ordering of the **upper** tails.

Crossing condition for stop-loss order

Theorem (Crossing condition for stop-loss order)

If for two r.v.'s, there is a real number c such that

$$\begin{aligned} F_X(x) &\leq F_Y(x), \quad \text{for all } x < c, \\ F_X(x) &\geq F_Y(x), \quad \text{for all } x \geq c, \end{aligned}$$

and if also $\mathbb{E}[X] \leq \mathbb{E}[Y]$, then

$$X \preceq_{sl} Y.$$

Definition

Definition (Second degree stochastic dominance)

Two r.v.'s X and Y are ordered in the Second degree stochastic dominance sense, notation $X \preceq_{sst} Y$, if

$$\mathbb{E}\left[u(X)\right] \leq \mathbb{E}\left[u(Y)\right],$$

for all non-decreasing and **concave** functions u.

- The class $\mathcal{U}$ is:

$$\mathcal{U} = \{u | u \text{ is a non-decreasing and } \textbf{concave} \text{ utility function}\}.$$

- $\mathcal{U}$ is the class of all **risk-averse** decision makers.
- Interpretation:
  - If $X \preceq_{sl} Y$, all risk-averse decision makers will prefer gain Y over X.

Example: Put option

- X denotes the price of a stock (e.g. Apple) at some future date T (e.g. one year).
- Put Option
  - A **put option** with strike K and maturity T gives the buyer the right to sell the stock at time T for the price K.
  - The buyer will benefit from this product when the stock price decreases.
- At maturity, the buyer will receive a pay-off equal to:

$$\text{Pay-off at maturity} = \begin{cases} K - X, & \text{if } X \leq K \\ 0, & \text{if } X > K \end{cases}.$$

$$\overset{\text{notation}}{=} (K - X)_+$$

- The expected pay-off is given by:

$$\mathbb{E}\left[(K - X)_+\right].$$

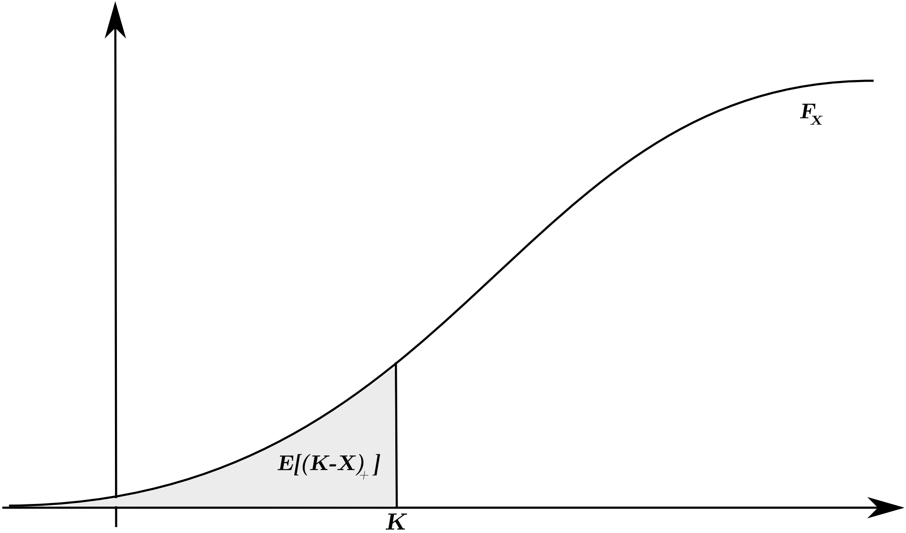
Measures for the lower tail

- Measure the lower tail using $\mathbb{E}\left[(K-X)_+\right]$.
- It can be proven that:

$$\mathbb{E}\left[(K-X)_+\right] = \int_{-\infty}^{K} F_X(x)\,\mathrm{d}x.$$

- Interpretation:
  - Lower tail at level K.
- lower tail transform:
  - $\lambda_X(x) = \mathbb{E}\left[(x-X)_+\right]$
  - A distribution is characterized by its lower tail transform:

$$\lambda'_X(x+) = F_X(x).$$



F_X
E[(K-X)+]
K

Losses versus gains

- 2nd degree stochastic dominance has no interpretation in terms of losses for risk-averse decision makers.
- Stop-loss and 2nd degree stochastic dominance:

$$X \preceq_{sst} Y \Leftrightarrow -Y \preceq_{sl} -X$$

- Alternative definition:

$$X \preceq_{sst} Y \Leftrightarrow \mathbb{E}\left[(K-X)_+\right] \geq \mathbb{E}\left[(K-Y)_+\right], \quad \text{for all } K \in \mathbb{R}.$$

- Interpretation:
  - The larger the lower tails of a gain, the less attractive this gain has to be considered.

Crossing condition

Definition (Crossing condition for sst-order)

If for two r.v.'s, there is a real number c such that

$$\begin{aligned} F_X(x) &\geq F_Y(x), \quad \text{for all } x < c, \\ F_X(x) &\leq F_Y(x), \quad \text{for all } x \geq c, \end{aligned}$$

and if also $\mathbb{E}[X] \leq \mathbb{E}[Y]$, then

$$X \preceq_{sst} Y.$$

Definition (Convex order)

Two r.v.'s X and Y are ordered in the convex order sense, notation $X \preceq_{cx} Y$ if

$$\mathbb{E}[X] = \mathbb{E}[Y] \quad \text{and} \quad \mathbb{E}[u(-X)] \geq \mathbb{E}[u(-Y)],$$

for all non-decreasing and **concave** functions u.

- $\mathcal{U}$ is the class of all risk-averse decision makers.
- Convex order gives the preferences of risk-averse decision makers between losses with the **same mean**.
- Interpretation:
  - A loss X is replaced by a less attractive r.v. Y, 'which is the same on average'.

The put-call parity

- Connection between upper and lower tail:

$$\mathbb{E}\left[(X-K)_+\right] = \mathbb{E}\left[(K-X)_+\right] + \mathbb{E}\left[X\right] - K.$$

- This expression is called the **put-call parity** and has (in a modified form) wide applications in option pricing theory.
- The put-call parity can be proven using the following expressions:

$$\begin{aligned}
\mathbb{E}\left[X\right] &= -\int_{-\infty}^{0} F_X(x)\mathrm{d}x + \int_{0}^{+\infty} \left(1 - F_X(x)\right)\mathrm{d}x, \\
\mathbb{E}\left[(X-K)_+\right] &= \int_{K}^{+\infty} \left(1 - F_X\left(x\right)\right)\mathrm{d}x, \\
\mathbb{E}\left[(K-X)_+\right] &= \int_{-\infty}^{K} F_X\left(x\right)\mathrm{d}x.
\end{aligned}$$

Ordered upper and lower tails

- Convex order and stop-loss order:

$$X \preceq_{cx} Y \Leftrightarrow \mathbb{E}[X] = \mathbb{E}[Y] \quad \text{and} \quad X \preceq_{sl} Y$$

- Convex order and upper tail transform:

$$X \preceq_{cx} Y \Leftrightarrow \begin{cases} \mathbb{E}[X] = \mathbb{E}[Y], \\ \mathbb{E}\left[(X-K)_+\right] \le \mathbb{E}\left[(Y-K)_+\right], \quad \text{for all } K, \end{cases}$$

  - Follows directly from the definition of stop-loss order.

- Convex order and lower tail transform:

$$X \preceq_{cx} Y \Leftrightarrow \begin{cases} \mathbb{E}[X] = \mathbb{E}[Y], \\ \mathbb{E}\left[(K-X)_+\right] \le \mathbb{E}\left[(K-Y)_+\right], \quad \text{for all } K, \end{cases}$$

  - Follows directly from the put-call parity.

Ordered upper and lower tails

- Convex order and second degree stochastic dominance:

$$X \preceq_{cx} Y \Leftrightarrow \mathbb{E}[X] = \mathbb{E}[Y] \quad \text{and} \quad Y \preceq_{sst} X$$

  - Follows directly from the definition of 2nd degree stochastic order.

- Convex order and lower and upper tail transforms:

$$X \preceq_{cx} Y \quad \Leftrightarrow \quad \begin{cases} X \preceq_{sl} Y, \\ Y \preceq_{sst} X, \end{cases} ,$$

$$\Leftrightarrow \quad \begin{cases} \mathbb{E}\left[(X-K)_+\right] \leq \mathbb{E}\left[(Y-K)_+\right], & \text{for all } K, \\ \mathbb{E}\left[(K-X)_+\right] \leq \mathbb{E}\left[(K-Y)_+\right], & \text{for all } K, \end{cases}$$

  - The proof of $\Rightarrow$ follows from previous relations.

Comparing variability of risks

- Convex order and convex/concave functions:

$$X \preceq_{cx} Y \Leftrightarrow \mathbb{E}\left[u(X)\right] \leq \mathbb{E}\left[u(Y)\right],$$

  for all convex functions u.

- $X \preceq_{cx} Y$ implies
  - Y has heavier upper tails than X,
  - Y has heavier lower tails than X.

- Convex order has an interpretation in terms of losses and gains.

$$X \preceq_{cx} Y \Leftrightarrow -X \preceq_{cx} -Y.$$

- R.v. X is 'less variable' than r.v. Y.

Theorem

If for two r.v.'s X and Y a real number c exists such that

$$\begin{aligned} F_X(x) &\leq F_Y(x) \text{ for all } x < c, \\ F_X(x) &\geq F_Y(x) \text{ for } x \geq c, \end{aligned} \tag{3}$$

and moreover $\mathbb{E}[X] = \mathbb{E}[Y]$, then $X \preceq_{cx} Y$.

Upper and lower tail transforms as building blocks

Lemma

For any $a \in \mathbb{R}$, $u(X)$ can be expressed as

$$u(X) = u(a) + u'(a)(X-a) + \int_{-\infty}^{a} u''(K)(K-X)_+ \, dK + \int_{a}^{+\infty} u''(K)(X-K)_+ \, dK. \tag{4}$$

- At a certain time, one has to pay the amount $u(X)$.
- The pay-off $u(X)$ can be decomposed using:
  - pay-off of call options: $(X-K)_+$,
  - pay-off of put options: $(K-X)_+$.
- Pay-offs of the form $(X-K)_+$ and $(K-X)_+$ are the building blocks for more complex pay-offs.

[3] see Carr & Madan (2001) and Cheung, Dhaene, Kukush & Linders (2013)

An important lemma

Lemma

For a r.v. X, we can decompose $\mathbb{E}\left[u\left(X\right)\right]$ as

$$\begin{aligned}\mathbb{E}\left[u\left(X\right)\right] =& u\left(\mathbb{E}\left[X\right]\right) + \int_{-\infty}^{\mathbb{E}[X]} u''\left(K\right)\mathbb{E}\left[\left(K-X\right)_{+}\right] dK \\ &+ \int_{\mathbb{E}[X]}^{+\infty} u''\left(K\right)\mathbb{E}\left[\left(X-K\right)_{+}\right] dK.\end{aligned}$$

- For a risk-averse decision maker, the expected utility $\mathbb{E}\left[u\left(X\right)\right]$ can be written as

  the utility of $\mathbb{E}[X]$ + portion of the tails.

- If $u''(x) \leq 0$, we immediately find Jensen's inequality:

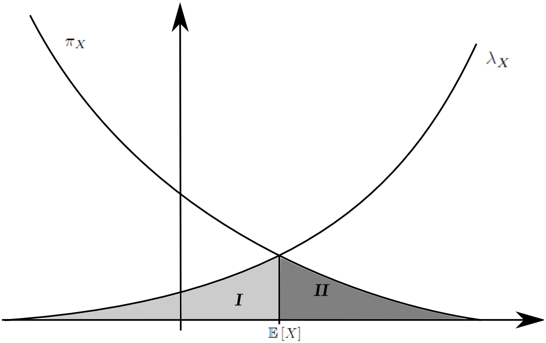
$$\mathbb{E}\left[u\left(X\right)\right] \leq u\left(\mathbb{E}\left[X\right]\right).$$

Example: Variance

- Variance decomposition formula
  - Take $u(x) = (x - \mathbb{E}[X])^2$, then:

$$\frac{1}{2}\mathrm{Var}[X] = \int_{-\infty}^{\mathbb{E}[X]} \mathbb{E}\left[(K - X)_+\right] \mathrm{d}K + \int_{\mathbb{E}[X]}^{+\infty} \mathbb{E}\left[(X - K)_+\right] \mathrm{d}K.$$

- Graphical representation:



- Convex ordered r.v.'s:
  - We can prove that $X \preceq_{cx} Y$, then

$$\int_{-\infty}^{\infty} \left(\mathbb{E}\left[(Y-K)_+ \right] - \mathbb{E}\left[(X-K)_+ \right] \right) \mathrm{d}K = \frac{1}{2} \left(\mathrm{Var}\left[Y\right] - \mathrm{Var}\left[X\right] \right).$$

  - Comparing variances is meaningful when comparing SL-premiums of convex ordered r.v.'s.
  - If $X \preceq_{cx} Y$, then $\mathrm{Var}[X] \leq \mathrm{Var}[Y]$.
- The following statements are equivalent:
  - $X \preceq_{cx} Y$ and $\mathrm{Var}[Y] = \mathrm{Var}[X]$
  - $X \stackrel{\mathrm{d}}{=} Y$.

Comparing expected utilities

Theorem

Consider the r.v.'s X and Y. Let u be a strictly concave function such that $\mathbb{E}\left[u\left(Y\right)\right]$ is finite.
Then

$$X \preceq_{cx} Y \text{ and } \mathbb{E}\left[u\left(X\right)\right] = \mathbb{E}\left[u\left(Y\right)\right] \tag{5}$$

is equivalent with

$$X \overset{\mathrm{d}}{=} Y. \tag{6}$$

- The function u has to have an absolutely continuous derivative to ensure all integrals are well-defined.
- The results hold for concave/convex twice differentiable utility functions.
- The condition $\mathbb{E}\left[u\left(X\right)\right] = \mathbb{E}\left[u\left(Y\right)\right]$ can be replaced by $\mathbb{E}\left[u\left(-X\right)\right] = \mathbb{E}\left[u\left(-Y\right)\right]$.

Comparing variances

- Consider a decision maker with a strictly concave and twice differentiable utility function:

$$u''(x) < 0.$$

  - The decision maker values X using the **whole** distribution.
- He has to choose between two convex ordered gains:

$$X \preceq_{cx} Y.$$

- If he is indifferent between X and Y
  - X and Y are equal, in distribution;
  - any other decision maker will be indifferent.

6 – References

- Barrieu, P. M. & Scandolo, G. (2013) ‘Assessing .nancial model risk’, Technical report. Available at SSRN: http://ssrn.com/abstract=2284101.
- Carr, P. & Madan, D. (2001) ‘Towards a theory of volatility trading’, in ‘Option Pricing, Interest Rates and Risk Management’, Cambridge University Press, pp. 458–476.
- Cheung, K. C., Dhaene, J., Kukush, A. & Linders, D. (2013), ‘Ordered random vectors and equality in distribution’, Scandinavian Actuarial Journal.
- Von Neumann, J. & Morgenstern, O. (1947), ‘Theory of games and economic behavior’, Princeton University Press. 2nd edition.
- Schoutens, W., Simons, E. & Tistaert, J. (2004) ‘A perfect calubration! Now what?’ Wilmott Magazine, March 2004.
- Yaari, M. E. (1987), ‘The dual theory of choice under risk’, Econometrica 55(1), 95 – 115.