Lecture 1: Exponential Utility Functions

Daniël Linders*

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1 Exponential utility and insurance premium

An insurer having initial wealth W and utility function u is said to have exponential utility if the function u can be expressed as follows:

$$u(x) = -\alpha e^{-\alpha x}.$$

Consider a non-negative risk X. We first determine the minimal premium P for which the insurer is willing to insure the risk X. This premium will be a function of the parameter α and we denote it as $P_X(\alpha)$. It is the solution of the following equation:

$$u(W) = \mathbb{E}\left[u\left(W + P - X\right)\right].$$

This equation can be solved for P as follows:

$$\begin{aligned} u(W) &= \mathbb{E}\left[u\left(W + P - X\right)\right] \\ \leftrightarrow -\alpha \mathrm{e}^{-\alpha W} &= \mathbb{E}\left[-\alpha \mathrm{e}^{-\alpha(W+P-X)}\right] \\ \leftrightarrow \mathrm{e}^{-\alpha W} &= \mathrm{e}^{-\alpha W}\mathrm{e}^{-\alpha P}\mathbb{E}\left[\mathrm{e}^{\alpha X}\right] \\ \leftrightarrow \mathrm{e}^{\alpha P} &= \mathbb{E}\left[\mathrm{e}^{\alpha X}\right]. \end{aligned}$$

We find the following premium $P_X(\alpha)$ for the risk X:

$$P_X(\alpha) = \frac{1}{\alpha}\log \mathbb{E}\left[\mathrm{e}^{\alpha X}\right].$$

This premium is independent of the initial wealth W.

The risk-aversion for an insurer having exponential utility with parameter α is constant and given by:

$$r(x) = \frac{-u''(x)}{u'(x)} = \frac{\alpha^3 \mathrm{e}^{-\alpha x}}{\alpha^2 \mathrm{e}^{-\alpha x}} = \alpha.$$

*University of Illinois & KU Leuven. Email: daniel.linders@kuleuven.be

2 Risk aversion and the insurance premium

The insurance premium $P_X(\alpha)$ is increasing in α. This can be proven as follows. Consider $0 < \alpha < \gamma$. Define the function v as $v(x) = x^{\alpha/\gamma}$. One can verify in a straightforward way that $v'(x) = \frac{\alpha}{\gamma} x^{\alpha/\gamma - 1}$ and $v''(x) = \frac{\alpha}{\gamma}\left(\frac{\alpha}{\gamma} - 1\right) x^{\alpha/\gamma - 2}$. Then we find that $v''(x) < 0$ and thus v is strictly concave. From Jensen's inequality, we find:

$$\mathbb{E}\left[v(Y)\right] < v\left(\mathbb{E}\left[Y\right]\right).$$

Define $Y = \mathrm{e}^{\gamma X}$. Then $v(Y) = \left(\mathrm{e}^{\gamma X}\right)^{\alpha/\gamma} = \mathrm{e}^{\alpha X}$. We can then prove the following inequality:

$$\begin{aligned}
\left(\mathbb{E}\left[\mathrm{e}^{\gamma X}\right]\right)^{\alpha} &= \left(\left(\mathbb{E}\left[\underbrace{\mathrm{e}^{\gamma X}}_{=Y}\right]\right)^{\alpha/\gamma}\right)^{\gamma} \\
&= v\left(\mathbb{E}\left[Y\right]\right)^{\gamma} \\
&> \mathbb{E}\left[v(Y)\right]^{\gamma} \\
&= \left(\mathbb{E}\left[\mathrm{e}^{\alpha X}\right]\right)^{\gamma}.
\end{aligned}$$

Taking the logarithms on both sides results in the following inequality:

$$P_X(\alpha) < P_X(\gamma).$$

We conclude that a higher risk aversion parameter results in a higher premium, i.e. the insurer requires a higher premium in order to take over the risk X from the insured.

Take α very small and remember the following Taylor expansions:

$$\begin{aligned}
\mathrm{e}^{\alpha x} &= 1 + \alpha x + \ldots \\
\log(1 + x) &= x + \ldots.
\end{aligned}$$

The premium $P_X(\alpha)$ can be approximated as follows:

$$\begin{aligned}
P_X(\alpha) &\approx \frac{1}{\alpha}\log\left(1 + \alpha\mathbb{E}[X]\right) \\
&\approx \frac{1}{\alpha}\alpha\mathbb{E}[X] \\
&= \mathbb{E}[X].
\end{aligned}$$

We conclude:

$$\lim_{\alpha \to 0} P_X(\alpha) = \mathbb{E}[X].$$

The insurance premium is increasing in the risk aversion. Hence, the expectation $\mathbb{E}[X]$ is always a lower bound for the insurance premium. The insurer is willing to take over the risk X for a premium equal to the expectation if he has no risk aversion, i.e. the insurer is risk neutral.

3 Aggregating exponential premiums

Consider an insurer with exponential utility function $u(x) = -\alpha e^{-\alpha x}$. The risks $X_1, X_2, \ldots, X_n$ are independent and they all have the same distribution as the r.v. X. So $X_i \overset{\mathrm{d}}{=} X$, for each $i = 1, 2, \ldots, n$. The aggregated loss S is equal to

$$S = X_1 + X_2 + \ldots + X_n.$$

The minimal premium the insurer wants to receive for insuring S is denoted by $P_S(\alpha)$. This premium $P_S(\alpha)$ satisfies the equation

$$\mathbb{E}\left[u\left(W + P_S(\alpha) - S\right)\right] = u(W).$$

We find that

$$P_S(\alpha) = \frac{1}{\alpha} \log \mathbb{E}\left[e^{\alpha S}\right].$$

If we use that the X_i are independent, we can write

$$\begin{aligned}
P_S(\alpha) &= \frac{1}{\alpha} \log \mathbb{E}\left[e^{\alpha(X_1+X_2+\ldots X_n)}\right] \\
&= \frac{1}{\alpha} \log \mathbb{E}\left[e^{\alpha X_1+\alpha X_2+\ldots \alpha X_n}\right] \\
&= \frac{1}{\alpha} \log\left(\mathbb{E}\left[e^{\alpha X_1}\right]\mathbb{E}\left[e^{\alpha X_2}\right]\ldots\mathbb{E}\left[e^{\alpha X_n}\right]\right) \\
&= \frac{1}{\alpha}\left(\log \mathbb{E}\left[e^{\alpha X_1}\right] + \log \mathbb{E}\left[e^{\alpha X_2}\right] + \ldots \log \mathbb{E}\left[e^{\alpha X_n}\right]\right) \\
&= n\frac{1}{\alpha} \log \mathbb{E}\left[e^{\alpha X}\right] \\
&= nP_X(\alpha)
\end{aligned}$$

If each insured pays his own premium $P_X(\alpha)$ for insuring his risk X_i, the insurer collects enough money to cover the insurance premium for S.

Define the r.v. Y as

$$Y = \frac{\sum_{i=1}^n X_i}{n}.$$

The premium $P_Y(\alpha)$ is given by

$$P_Y(\alpha) = \frac{1}{\alpha} \log \mathbb{E}\left[e^{\alpha Y}\right].$$

We can write

$$\begin{aligned}
P_Y(\alpha) &= \frac{1}{\alpha} \log \mathbb{E}\left[e^{\alpha \frac{\sum_{i=1}^n X_i}{n}}\right] \\
&= \frac{1}{\alpha} \log \mathbb{E}\left[e^{\frac{\alpha}{n}X_1+\frac{\alpha}{n}X_2+\ldots\frac{\alpha}{n}X_n}\right] \\
&= \frac{1}{\alpha} \log\left(\mathbb{E}\left[e^{\frac{\alpha}{n}X_1}\right]\mathbb{E}\left[e^{\frac{\alpha}{n}X_2}\right]\ldots\mathbb{E}\left[e^{\frac{\alpha}{n}X_n}\right]\right) \\
&= n\frac{1}{\alpha} \log \mathbb{E}\left[e^{\frac{\alpha}{n}X}\right] \\
&= P_X\left(\frac{\alpha}{n}\right).
\end{aligned}$$

Note that the premium $P_X(\alpha)$ is increasing in α, which means that

$$\mathbb{E}[X] \leq P_Y(\alpha) \leq P_X(\alpha).$$

The insurer can aggregate all the losses and divide it equally over its policy holders. Then each policy holder pays the loss Y. Therefore, it is sufficient to ask the premium $P_Y(\alpha)$. If n is large, α/n tends to zero. Taking into account $\mathbb{E}[Y] = \mathbb{E}[X]$, we find that $P_Y(\alpha) \approx \mathbb{E}[X]$, if n is large.

The random vector $(X_1, X_2, \ldots, X_n)$ contains independent random variables. Knowledge about the realization of the first risk X_1, does not give any information about the realization for X_2. The random vector $(Y_1, Y_2, \ldots, Y_n)$, where $Y_i = Y$, contains random variables which are dependent. Moreover, they are extremely positive dependent in that knowing Y_1 means the realization of all other random variables is known. Because the random losses Y_i are not independent anymore, one cannot add them to determine the aggregate premium.