Lecture 2: Workings on convex order and mututal exclusive risks

Daniël Linders*

Version: September 23, 2019

1 Sum of independent risks

Consider the random variables X and Y, which have the same distribution:

$$\begin{aligned}\mathbb{P}[X=0] &= \mathbb{P}[X=100]=\frac{1}{2},\\ \mathbb{P}[Y=0] &= \mathbb{P}[Y=100]=\frac{1}{2}.\end{aligned}$$

The bivariate distribution $\left(X^{\perp},Y^{\perp}\right)$ has the same marginal distributions than (X,Y) but the components are mutually independent. We call $\left(X^{\perp},Y^{\perp}\right)$ the independent modification of the random vector (X,Y). If the r.v's $X^{\perp}$ and $Y^{\perp}$ are independent, we have that:

$$\begin{aligned}\mathbb{P}\left[X^{\perp}=0,Y^{\perp}=0\right] &= \mathbb{P}\left[X^{\perp}=0\right]\mathbb{P}\left[Y^{\perp}=0\right]=\frac{1}{4},\\ \mathbb{P}\left[X^{\perp}=0,Y^{\perp}=100\right] &= \mathbb{P}\left[X^{\perp}=0\right]\mathbb{P}\left[Y^{\perp}=100\right]=\frac{1}{4},\\ \mathbb{P}\left[X^{\perp}=100,Y^{\perp}=0\right] &= \mathbb{P}\left[X^{\perp}=100\right]\mathbb{P}\left[Y^{\perp}=0\right]=\frac{1}{4},\\ \mathbb{P}\left[X^{\perp}=100,Y^{\perp}=100\right] &= \mathbb{P}\left[X^{\perp}=100\right]\mathbb{P}\left[Y^{\perp}=100\right]=\frac{1}{4}.\end{aligned}$$

The possible outcomes of $S^{\perp}$ are $0, 100$ or 200. The corresponding probabilities are given by:

$$\begin{aligned}\mathbb{P}\left[S^{\perp}=0\right] &= \mathbb{P}[X=0,Y=0]=\frac{1}{4}\\ \mathbb{P}\left[S^{\perp}=100\right] &= \mathbb{P}[X=0,Y=100]+\mathbb{P}[X=100,Y=0]=\frac{1}{2}\\ \mathbb{P}\left[S^{\perp}=200\right] &= \mathbb{P}[X=100,Y=100]=\frac{1}{4}.\end{aligned}$$

*Univeristy of Illinois & KU Leuven. Email: daniel.linders@kuleuven.be

The cdf $F_{S^\perp}$ is defined as $F_{S^\perp}(x) = \mathbb{P}\left[S^\perp \leq x\right]$. From the probability distribution function, we can deduce the cdf:

$$F_{S^\perp}(x) = \begin{cases} 0, & \text{if } x < 0; \\ 1/4, & \text{if } 0 \leq x < 100; \\ 3/4, & \text{if } 100 \leq x < 200; \\ 1, & \text{if } 200 \leq x. \end{cases}$$

The stop-loss premium $\mathbb{E}\left[(S^\perp - K)_+\right]$ can be written as follows:

$$\mathbb{E}\left[(S^\perp - K)_+\right] = \int_K^{+\infty} (1 - F_{S^\perp}(x))\, dx.$$

If $K \geq 200$, then it is straightforward to show that $\mathbb{E}\left[(S^\perp - K)_+\right] = 0$. Take $K \in [100, 200)$. Then:

$$\begin{aligned} \mathbb{E}\left[(S^\perp - K)_+\right] &= \int_K^{200} (1 - F_{S^\perp}(x))\, dx \\ &= \int_K^{200} \left(1 - \frac{3}{4}\right) dx \\ &= \frac{200 - K}{4}. \end{aligned}$$

If $K \in [0, 100)$. Then:

$$\begin{aligned} \mathbb{E}\left[(S^\perp - K)_+\right] &= \int_K^{200} (1 - F_{S^\perp}(x))\, dx \\ &= \int_K^{100} (1 - F_{S^\perp}(x))\, dx + \int_{100}^{200} (1 - F_{S^\perp}(x))\, dx \\ &= \int_K^{100} (1 - F_{S^\perp}(x))\, dx + \frac{100}{4} \\ &= \int_K^{100} \left(1 - \frac{1}{4}\right) dx + 25 \\ &= \frac{3}{4}(100 - K) + 25. \end{aligned}$$

Combining all results in:

$$\mathbb{E}\left[(S^\perp - K)_+\right] = \begin{cases} 100 - K, & \text{if } K < 0; \\ 25 + (100 - K)\frac{3}{4}, & \text{if } 0 \leq K < 100; \\ \frac{200-K}{4}, & \text{if } 100 \leq K < 200; \\ 0, & \text{if } 200 \leq K. \end{cases} \tag{1}$$

2 Sum of mutual exclusive risks

The bivariate distribution (X^m, Y^m) has the same marginal distributions than (X, Y), but the components are mutually exclusive . We call (X^m, Y^m) the mutual exclusive modification of the random vector (X, Y). If the r.v.'s X^m and Y^m are mutually exclusive, we have that

$$\mathbb{P}\left[X^m = 100, Y^m = 100\right] = 0.$$

Using the information about the marginal probabilities, we can solve for the remaining bivariate probabilities. For example, $\mathbb{P}[X^m = 100, Y^m = 0]$ follows from:

$$\underbrace{\mathbb{P}[X^m = 100]}_{=1/2} = \mathbb{P}[X^m = 100, Y^m = 0] + \underbrace{\mathbb{P}[X^m = 100, Y^m = 100]}_{=0},$$

from which we find that $\mathbb{P}[X^m = 100, Y^m = 0] = \frac{1}{2}$. A similar reasoning results in the following probabilities:

$$\begin{aligned}
\mathbb{P}[X^m = 0, Y^m = 0] &= 0 \\
\mathbb{P}[X^m = 0, Y^m = 100] &= \frac{1}{2} \\
\mathbb{P}[X^m = 100, Y^m = 0] &= \frac{1}{2} \\
\mathbb{P}[X^m = 100, Y^m = 100] &= 0
\end{aligned}$$

The mutual exclusive sum S^m is defined qs:

$$S^m = X^m + Y^m.$$

The derivation of the cdf of S^m is similar to the independent case. First, we determine the probability distribution:

$$\begin{aligned}
\mathbb{P}[S^m = 0] &= \mathbb{P}[X^m = 0, Y^m = 0] = 0 \\
\mathbb{P}[S^m = 100] &= \mathbb{P}[X^m = 0, Y^m = 100] + \mathbb{P}[X^m = 100, Y^m = 0] = 1 \\
\mathbb{P}[S^m = 200] &= \mathbb{P}[X^m = 100, Y^m = 100] = 0.
\end{aligned}$$

The stop-loss premium $\mathbb{E}[(S^m - K)_+]$ is then given by

$$\mathbb{E}[(S^m - K)_+] = \begin{cases} 100 - K, & \text{if } K < 100; \\ 0, & \text{if } 100 \leq K. \end{cases} \tag{2}$$

The expected values of both sums are equal. The convex order follows immediately if we compare (1) with (2):

$$S^m \preceq_{cx} S^{\perp}.$$

In case the risks are mutual exclusive, they move 'in the opposite direction', which results in a safer distribution for the aggregate risk. Note also that Var$[S^m] = 0$ and the vector (X^m, Y^m) is said to be complete mixable.

3 A minimal element in convex order

Consider the random vector (X, Y) with marginals given above but the dependence structure is not specified. We will show that

$$\mathbb{E}[(S^m - K)_+] \leq \mathbb{E}[(S - K)_+], \text{ for all } K,$$

where

$$S = X + Y.$$

Because X and Y are two-point distributions, we have that $S \in \{0, 100, 200\}$. Moreover, $\mathbb{E}[S] = \mathbb{E}[S^m] = 100$ and therefore

$$\mathbb{E}[(S-K)_+] = 100 - K, \text{ for } K \leq 0.$$

Because the stop-loss curve of S has to be convex, it immediately follows that

$$\mathbb{E}[(S-K)_+] \geq 100 - K, \text{ for all } K,$$

from which we find that

$$\mathbb{E}[(S^m - K)_+] \leq \mathbb{E}[(S-K)_+], \text{ for all } K,$$

or equivalently

$$S^m \preceq_{cx} S.$$

The mutual exclusive modification (X^m, Y^m) is said to be minimal in convex order, in the sense that any random vector having the same marginals but different dependence structure has a sum which exceeds S^m in convex order. Because of this property, mutual exclusivity is sometimes referred to as the safes dependence structure, because aggregating mutual exclusive risks corresponds with a maximal diversification effect.

4 Remarks: The Frechet space

The random vectors $\left(X^\perp, Y^\perp\right)$ and (X^m, Y^m) have the same marginal distributions, but a different dependence structure. It is shown in this simple example that the **dependence structure** can have an important impact on the distribution of the aggregate risk. Therefore, modeling and measuring dependence is crucial when considering problems consisting of dependent random variables such as Solvency calculation, portfolio selection, derivative pricing, etc.

The Frechet space $\mathcal{R}_2(F_X, F_Y)$ is defined as follows:

$$\mathcal{R}_2(F_X, F_Y) = \{(V, W) |\, F_V \equiv F_X, F_W \equiv F_Y\}.$$

The set $\mathcal{R}_2(F_X, F_Y)$ contains all random vectors (V, W) having the same marginal distributions than the original random vector (X, Y). Two members of the Frechet space are the independent and the mutual exclusive modification of (X, Y):

$$\left(X^\perp, Y^\perp\right), (X^m, X^m) \in \mathcal{R}_2(F_X, F_Y).$$

In the sequel of this course we will search for other important elements in the Frechet space. Moreover, we consider parametric models to generate members of the Frechet space.

We showed that the two elements in the Frechet space are ordered in the convex order sense: $S^m \preceq_{cx} S^\perp$. One can think about the following questions:

- Does there exist a random vector $(V, W) \in \mathcal{R}_2(F_X, F_Y)$ such that $V + W \preceq_{cx} S^m$?

- Does there exist a random vector (X^u, Y^u) such that for any $(V, W) \in \mathcal{R}_2(F_X, F_Y)$ we have that $V + W \preceq_{cx} S^u$?
- Does the random vector (X^u, Y^u) belong to the Frechet space?
- ...

In the following chapters we will answer these questions!