Chapter 4: Workings on sums of comonotonic random variables

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1 The lognormal distribution

The random variable X is said to be lognormal distribution with parameters μ and σ, notation $X \sim LN(\mu, \sigma^2)$ if

$$\log X \sim N(\mu, \sigma^2).$$

If we denote by Y a normal random variable with mean μ and variance σ^2, we have that

$$X \overset{d}{=} \mathrm{e}^Y.$$

Note that the moment generating function of a normal distribution is given in closed form:

$$\mathbb{E}\left[\mathrm{e}^{tY}\right] = \mathrm{e}^{t\mu + \frac{1}{2}t^2\sigma^2}.$$

The mean of X can then be derived as follows:

$$\begin{aligned} \mathbb{E}[X] &= \mathbb{E}\left[\mathrm{e}^Y\right] \\ &= \mathrm{e}^{\mu + \frac{\sigma^2}{2}} \end{aligned}$$

The variance of X follows from:

$$\begin{aligned} \text{Var}[X] &= \mathbb{E}\left[X^2\right] - \mathbb{E}\left[X\right]^2 \\ &= \mathbb{E}\left[\mathrm{e}^{2Y}\right] - \left(\mathrm{e}^{\mu + \frac{\sigma^2}{2}}\right)^2 \\ &= \mathrm{e}^{2\mu + \frac{4\sigma^2}{2}} - \mathrm{e}^{2\mu + \frac{2\sigma^2}{2}} \\ &= \mathrm{e}^{2\mu + \sigma^2}\left(\mathrm{e}^{\sigma^2} - 1\right) \end{aligned}$$

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2 Correlation between non-comonotonic lognormal distributions

Consider the random variables X_1 and X_2 which are both lognormal distributed:

$$X_i \sim LN(\mu_i, \sigma_i^2), \text{ for } i = 1, 2.$$

Define the normal distributed random variables Y_i as follows: $Y_i = \log X_i$ and assume that the correlation between these normal random variables is given by ρ:

$$\text{Corr}\left[Y_1, Y_2\right] = \rho.$$

Moreover, it is assumed that the random vector (Y_1, Y_2) is bivariate normal, which implies that any linear combination of Y_1 and Y_2 has again a normal distribution.

The correlation $\text{Corr}[X_1, X_2]$ between the *lognormal* random variables can be calculated as follows:

$$\begin{aligned}\text{Corr}\left[X_1, X_2\right] &= \frac{\text{Cov}\left[X_1, X_2\right]}{\sqrt{\text{Var}\left[X_1\right]\text{Var}\left[X_2\right]}} \\ &= \frac{\mathbb{E}\left[X_1 X_2\right] - \mathbb{E}\left[X_1\right]\mathbb{E}\left[X_2\right]}{\sqrt{\text{Var}\left[X_1\right]\text{Var}\left[X_2\right]}}\end{aligned}$$

We still have to determine $\mathbb{E}\left[X_1 X_2\right]$:

$$\begin{aligned}\mathbb{E}\left[X_1 X_2\right] &= \mathbb{E}\left[\mathrm{e}^{Y_1}\mathrm{e}^{Y_2}\right] \\ &= \mathbb{E}\left[\mathrm{e}^{Y_1+Y_2}\right].\end{aligned}$$

Note that $Y_1 + Y_2$ has again a normal distribution because (Y_1, Y_2) is assumed to be bivariate normal:

$$Y_1 + Y_2 \overset{\mathrm{d}}{=} N\left(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2\right).$$

If $Y_1 + Y_2$ has a normal distribution, $X_1 X_2 = \mathrm{e}^{Y_1+Y_2}$ has a lognormal distribution. Using the expression for the expectation of a lognormal r.v. derived in the previous section, we find that

$$\begin{aligned}\mathbb{E}\left[X_1 X_2\right] &= \mathbb{E}\left[\mathrm{e}^{Y_1+Y_2}\right] \\ &= \mathrm{e}^{\mu_1+\mu_2+\frac{1}{2}\left(\sigma_1^2+\sigma_2^2+2\rho\sigma_1\sigma_2\right)} \\ &= \mathrm{e}^{\mu_1+\mu_2+\frac{1}{2}\left(\sigma_1^2+\sigma_2^2\right)+\rho\sigma_1\sigma_2}.\end{aligned}$$

Combining the expression for $\mathbb{E}\left[X_1 X_2\right]$ and $\mathbb{E}\left[X_i\right]$, we can write the covariance as

$$\begin{aligned}\text{Cov}\left[X_1, X_2\right] &= \mathbb{E}\left[X_1 X_2\right] - \mathbb{E}\left[X_1\right]\mathbb{E}\left[X_2\right] \\ &= \mathrm{e}^{\mu_1+\mu_2+\frac{1}{2}\left(\sigma_1^2+\sigma_2^2\right)+\rho\sigma_1\sigma_2} - \mathrm{e}^{\mu_1+\frac{1}{2}\sigma_1^2}\mathrm{e}^{\mu_2+\frac{1}{2}\sigma_2^2} \\ &= \mathrm{e}^{\mu_1+\mu_2+\frac{1}{2}\left(\sigma_1^2+\sigma_2^2\right)}\left(\mathrm{e}^{\rho\sigma_1\sigma_2} - 1\right).\end{aligned}$$

The correlation becomes:

$$\begin{aligned}
\text{Corr}\left[X_1, X_2\right] &= \frac{\text{Cov}\left[X_1, X_2\right]}{\sqrt{\text{Var}\left[X_1\right]\text{Var}\left[X_2\right]}} \\
&= \frac{\mathrm{e}^{\mu_1+\mu_2+\frac{1}{2}\left(\sigma_1^2+\sigma_2^2\right)}\left(\mathrm{e}^{\rho\sigma_1\sigma_2}-1\right)}{\sqrt{\mathrm{e}^{2\mu_1+\sigma_1^2}\left(\mathrm{e}^{\sigma_1^2}-1\right)e^{2\mu_2+\sigma_2^2}\left(\mathrm{e}^{\sigma_2^2}-1\right)}} \\
&= \frac{\mathrm{e}^{\mu_1+\mu_2+\frac{1}{2}\left(\sigma_1^2+\sigma_2^2\right)}\left(\mathrm{e}^{\rho\sigma_1\sigma_2}-1\right)}{\mathrm{e}^{\mu_1+\frac{1}{2}\sigma_1^2}\sqrt{\mathrm{e}^{\sigma_1^2}-1}\mathrm{e}^{\mu_2+\frac{1}{2}\sigma_2^2}\sqrt{\mathrm{e}^{\sigma_2^2}-1}} \\
&= \frac{\mathrm{e}^{\rho\sigma_1\sigma_2}-1}{\sqrt{\mathrm{e}^{\sigma_1^2}-1}\sqrt{\mathrm{e}^{\sigma_2^2}-1}}
\end{aligned}$$

3 Correlation between comonotonic lognormal random variables

The comonotonic modification (X_1^c, X_2^c) of the random vector (X_1, X_2) is given by:

$$(X_1^c, X_2^c) = \left(F_{X_1}^{-1}(U), F_{X_2}^{-1}(U)\right).$$

We will derive the correlation between the comonotonic random variables. A first approach is to use the expression obtained above and take the limit for $\rho \to 1$. However, below we give a more formal derivation of this result.

We have that $X_i \overset{d}{=} \mathrm{e}^{Y_i} \overset{d}{=} \mathrm{e}^{\mu_i+\sigma_i\Phi^{-1}(U)}$. Then:

$$F_{X_i}^{-1}(p) = \mathrm{e}^{\mu_i+\sigma_i\Phi^{-1}(p)}.$$

Then, we can write the comonotonic random variable (X_1^c, X_2^c) as:

$$(X_1^c, X_2^c) = \left(\mathrm{e}^{\mu_1+\sigma_1\Phi^{-1}(U)}, \mathrm{e}^{\mu_2+\sigma_2\Phi^{-1}(U)}\right).$$

The correlation $\text{Corr}\left[X_1^c, X_2^c\right]$ is given by

$$\begin{aligned}
\text{Corr}\left[X_1^c, X_2^c\right] &= \frac{\text{Cov}\left[X_1^c, X_2^c\right]}{\sqrt{\mathrm{e}^{2\mu_1+\sigma_1^2}\left(e^{\sigma_1^2}-1\right)\mathrm{e}^{2\mu_2+\sigma_2^2}\left(e^{\sigma_2^2}-1\right)}} \\
&= \frac{\text{Cov}\left[X_1^c, X_2^c\right]}{\mathrm{e}^{\mu_1+\mu_2}\mathrm{e}^{\frac{\sigma_1^2+\sigma_2^2}{2}}\sqrt{\left(e^{\sigma_1^2}-1\right)\left(e^{\sigma_2^2}-1\right)}} \qquad (1)
\end{aligned}$$

The covariance can be calculated as follows:

$$\begin{aligned}
\text{Cov}[X_1^c, X_2^c] &= \mathbb{E}\left[F_{X_1}^{-1}(U)F_{X_2}^{-1}(U)\right] - \mathbb{E}\left[F_{X_1}^{-1}(U)\right]\mathbb{E}\left[F_{X_2}^{-1}(U)\right] \\
&= \mathbb{E}\left[\mathrm{e}^{\mu_1+\sigma_1\Phi^{-1}(U)}\mathrm{e}^{\mu_2+\sigma_2\Phi^{-1}(U)}\right] - \mathrm{e}^{\mu_1+\frac{1}{2}\sigma_1^2}\mathrm{e}^{\mu_2+\frac{1}{2}\sigma_2^2} \\
&= \mathbb{E}\left[\mathrm{e}^{\mu_1+\mu_2+(\sigma_1+\sigma_2)\Phi^{-1}(U)}\right] - \mathrm{e}^{\mu_1+\mu_2+\frac{1}{2}(\sigma_1^2+\sigma_2^2)} \qquad (2)
\end{aligned}$$

Plugging the Expression (2) in (1) gives

$$\mathrm{Corr}\left[X_1^c, X_2^c\right] = \frac{\mathbb{E}\left[\mathrm{e}^{(\sigma_1+\sigma_2)\Phi^{-1}(U)}\right] - \mathrm{e}^{\frac{1}{2}(\sigma_1^2+\sigma_2^2)}}{\mathrm{e}^{\frac{1}{2}(\sigma_1^2+\sigma_2^2)}\sqrt{\left(e^{\sigma_1^2}-1\right)\left(e^{\sigma_2^2}-1\right)}}.$$

Note that $\Phi^{-1}(U)$ is a standard normal random variable. Using the expression for its moment generating functions results in the following expression:

$$\mathbb{E}\left[\mathrm{e}^{(\sigma_1+\sigma_2)\Phi^{-1}(U)}\right] = \mathrm{e}^{\frac{1}{2}(\sigma_1+\sigma_2)^2},$$

yielding

$$\begin{aligned}
\mathrm{Corr}\left[X_1^c, X_2^c\right] &= \frac{\mathrm{e}^{\frac{1}{2}(\sigma_1+\sigma_2)^2} - \mathrm{e}^{\frac{1}{2}(\sigma_1^2+\sigma_2^2)}}{\mathrm{e}^{\frac{1}{2}(\sigma_1^2+\sigma_2^2)}\sqrt{\left(e^{\sigma_1^2}-1\right)\left(e^{\sigma_2^2}-1\right)}} \\
&= \frac{\mathrm{e}^{\frac{1}{2}(\sigma_1^2+\sigma_2^2+2\sigma_1\sigma_2)} - \mathrm{e}^{\frac{1}{2}(\sigma_1^2+\sigma_2^2)}}{\mathrm{e}^{\frac{1}{2}(\sigma_1^2+\sigma_2^2)}\sqrt{\left(e^{\sigma_1^2}-1\right)\left(e^{\sigma_2^2}-1\right)}} \\
&= \frac{\cancel{\mathrm{e}^{\frac{1}{2}(\sigma_1^2+\sigma_2^2)}}\mathrm{e}^{\sigma_1\sigma_2} - \cancel{\mathrm{e}^{\frac{1}{2}(\sigma_1^2+\sigma_2^2)}}}{\cancel{\mathrm{e}^{\frac{1}{2}(\sigma_1^2+\sigma_2^2)}}\sqrt{\left(e^{\sigma_1^2}-1\right)\left(e^{\sigma_2^2}-1\right)}}
\end{aligned}$$

So we find:

$$\mathrm{Corr}\left[X_1^c, X_2^c\right] = \frac{\mathrm{e}^{\sigma_1\sigma_2}-1}{\sqrt{\left(e^{\sigma_1^2}-1\right)\left(e^{\sigma_2^2}-1\right)}}$$