KU LEUVEN

Foundations of Quantitative Risk Measurement

Chapter 5: Yaari's Dual Theory

Jan Dhaene & Daniël Linders

November, 2019

The problem of choosing between risks

- Setting: A decision maker can choose between two (random) losses: X and Y.
- Question:

  What is the 'optimal' choice?
- Naive approach: Compare the expectations:

$$\text{Prefer loss } X \text{ over } Y \Leftrightarrow \mathbb{E}[X] \leq \mathbb{E}[Y].$$

The problem of choosing between risks

- **Remark 1**: What if $\mathbb{E}[X] = \mathbb{E}[Y]$?
  - Is the decision maker indifferent?
  - even if X and Y are behaving very differently?
- **Remark 2:** The devil is in the tails!
  - Decision makers are assumed to be risk-averse.
  - The main concern is to *avoid (extreme) losses*.

The problem of choosing between risks

- The most important insight from Chapter 1:
  - In order to understand which losses are preferred,
  - we have to take into account the risk-preferences of the decision maker.
- **Consequence 1:** Different decision makers make different choices.
  - Risk-preferences differ between decision makers.
- **Consequence 2:** We need to model qualitative preferences using quantitative models.
  - For example: in chapter 1 we used utility functions are used to model risk-preferences,
  - and expected utilities to order random losses.

The problem of choosing between risks

- Yaari's theory of choice under risk is an alternative for the expected utility theory.
    - Risk preferences of decision makers are captured in a *distortion function* g.
    - A decision maker always maximizes his *distorted wealth level*.
- Risk preferences: Distortion functions
    - instead of utility functions.
    - We keep the outcomes of the losses, but consider subjective probabilities.
- Risk ordering: Distorted wealth levels
    - instead of expected utility.
    - in Yaari's theory, we compare expectations, but under a distorted probability measure.

Introduction

- Distortion risk measures:
  - Generate coherent risk measures: See later chapter.
  - Reference: Dhaene et al (2006).
- Sublinear expectations:
  - A distorted expectation is a generalization of the classical expectation.
  - It is an expectation under distorted probability levels.
  - Under some continuity conditions, a distorted expectation is a classical expectation under a new probability measure.
  - Reference: Peng (2010).
- Bid-ask pricing:
  - A fundamental approach to justify bid-ask prices in asset pricing.
  - See: Madan & Schoutens (2016) 'Applied Conic Finance' (book).

- Consider the gains X and Y:

$$\mathbb{P}[X = 1] = 1 \quad \text{and} \quad \mathbb{P}[Y = x] = \begin{cases} 0.01, & x = 0; \\ 0.89, & x = 1; \\ 0.1, & x = 5. \end{cases}$$

- Consider the gains V and W:

$$\mathbb{P}[V = x] = \begin{cases} 0.89, & x = 0; \\ 0.11, & x = 1. \end{cases} \quad \text{and} \quad \mathbb{P}[W = x] = \begin{cases} 0.9, & x = 0; \\ 0.1, & x = 5. \end{cases}$$

- Empirical studies reveal that many people prefer X over Y, but W over V.

Introduction

- Expected utility maximizers who prefer X over Y, will also prefer V over W.
- However, if a decision maker prefers X over Y, but W over V, then he/she **can never be** expected utility maximizers.
- Paradox in expected utility theory.

Distortion function: definition

Definition

A distortion function is a non-decreasing function $g : [0,1] \to [0,1]$ such that $g(0) = 0$ and $g(1) = 1$.

- g is non-decreasing implies that g is *continuous* and *differentiable* on $[0,1]$, almost everywhere.
- A distortion function is said to be concave (convex) if it is concave (convex) on $[0,1]$ without jumps in 0 and 1.
- Concave and convex distortion functions are continuous on $(0,1)$.

Distortion function: definition

- Consider a random variable X. The tail probability $\overline{F}_X(x)$ is given by

$$\overline{F}_X(x) = \mathbb{P}\left[X > x\right].$$

- The function g is used to distort the (tail) probabilities:
  - If g is concave: $g\left(\overline{F}_X(x)\right) \geq \overline{F}_X(x)$.
  - If g is convex: $g\left(\overline{F}_X(x)\right) \leq \overline{F}_X(x)$.
  - Exercise: Prove these inequalities.

Distortion function: Example

- Consider a random variable X:

$$X \stackrel{d}{=} LN\left(\mu, \sigma^2\right).$$

- Tail probabilities:

$$\overline{F}_X(x) = \mathbb{P}[X > x] = \Phi\left(\frac{\mu - \ln x}{\sigma}\right), \tag{1}$$

where Φ is the cdf of a standard normal distribution.

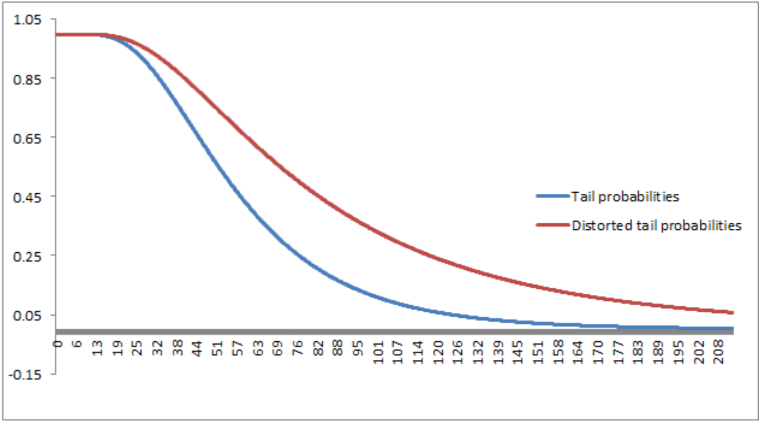
  - Exercise: Prove this equality.

- Distorted tail probabilities:
  - $g(x) = \sqrt{x}$.

$$g\left(\overline{F}_X(x)\right) = \sqrt{\Phi\left(\frac{\mu - \ln x}{\sigma}\right)}. \tag{2}$$

Distortion function: Example

- Parameters: $\mu = 4$ and $\sigma = 0.5$.
- Mean: $\mathbb{E}[X] = e^{\mu + \frac{\sigma^2}{2}} \approx 62$.



Definition

Definition

Consider a distortion function g. The distorted expectation of the r.v. X, notation $\rho_g[X]$, is

$$\rho_g[X] = -\int_{-\infty}^{0} \left[1 - g\left(\overline{F}_X(x)\right)\right] \mathrm{d}x + \int_{0}^{+\infty} g\left(\overline{F}_X(x)\right) \mathrm{d}x,$$

provided both integrals are finite.

- The functional ρ_g is called the *distortion risk measure* (with distortion function g).
- Both integrals are assumed to be finite, which implies that $\rho_g[X]$ is finite.

Distorted expectations and the distorted tail probabilities

- The expectation of X:

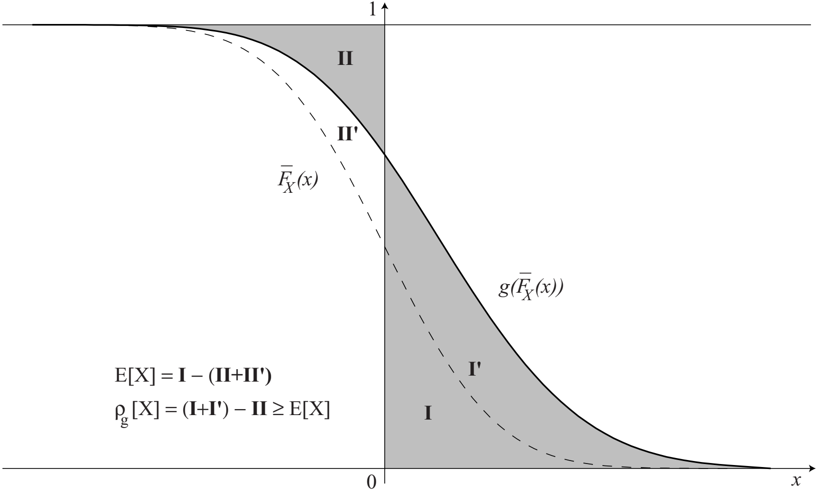
$$\mathbb{E}[X] = -\int_{-\infty}^{0} \left(1 - \overline{F}_X(x)\right) \mathrm{d}x + \int_{0}^{+\infty} \overline{F}_X(x) \mathrm{d}x.$$

- Distorted expectation of X:

$$\rho_g[X] = -\int_{-\infty}^{0} \left[1 - g\left(\overline{F}_X(x)\right)\right] \mathrm{d}x + \int_{0}^{+\infty} g\left(\overline{F}_X(x)\right) \mathrm{d}x.$$

- Interpretation:
  - If g is left continuous, then $g\left(\overline{F}(x)\right)$ is non-increasing and right continuous with values in $[0,1]$.
  - The distorted expectation $\rho_g[X]$ can be interpreted as an expectation of X, but the tail probabilities $\overline{F}_X(x)$ are replaced by the distorted tail probabilities $g\left(\overline{F}_X(x)\right)$.

graphical interpretation for concave g



Continuous distortion functions: Examples

- A distortion function is continuous if it is continuous on $[0,1]$.
  - Example
    - ⋆ Distortion function

$$g(q) = q, \quad \text{for } \ q \in [0,1].$$

    - ⋆ Corresponding distorted expectation

$$\rho_g[X] = \mathbb{E}[X].$$

    - ⋆ The distortion function g doesn't distort the probabilities.
- Concave/convex functions
  - If g is concave: $\rho_g[X] \geq \mathbb{E}[X]$.
  - If g is convex: $\rho_g[X] \leq \mathbb{E}[X]$.
  - Exercise: Prove these inequalities.

Properties

Theorem

For any distortion function g and any r.v.'s X and Y, the following properties hold:

1. *Positive homogeneity:*

$$\rho_g[aX] = a\rho_g[X], \quad a > 0;$$

2. *Translation invariance:*

$$\rho_g[X+b] = \rho_g[X] + b, \quad b \in \mathbb{R};$$

3. *Preserving stochastic dominance:*

$$\text{if } F_X(x) \geq F_Y(x), \quad \text{for all } x \in \mathbb{R} \Rightarrow \rho_g[X] \leq \rho_g[Y].$$

Right and left continuous distortion function: Example

- Consider the following r.c. distortion function

$$g(q) = \mathbb{I}\left(q \geq 1 - p\right), \quad 0 \leq q \leq 1.$$

- Corresponding distorted expectation:

$$\rho_g[X] = F_X^{-1+}(p).$$

- The condition that g is r.c. is essential:
  - Consider the l.c. distortion function

$$g(q) = \mathbb{I}\left(q > 1 - p\right), \quad 0 \leq q \leq 1.$$

  - Corresponding distorted expectation:

$$\rho_g[X] = F_X^{-1}(p).$$

Dual distortion function

Definition

The dual distortion function $\overline{g}$ of the distortion function g is

$$\overline{g}(q) = 1 - g(1-q).$$

- $\overline{g}$ is again a distortion function.
- right and left continuous functions

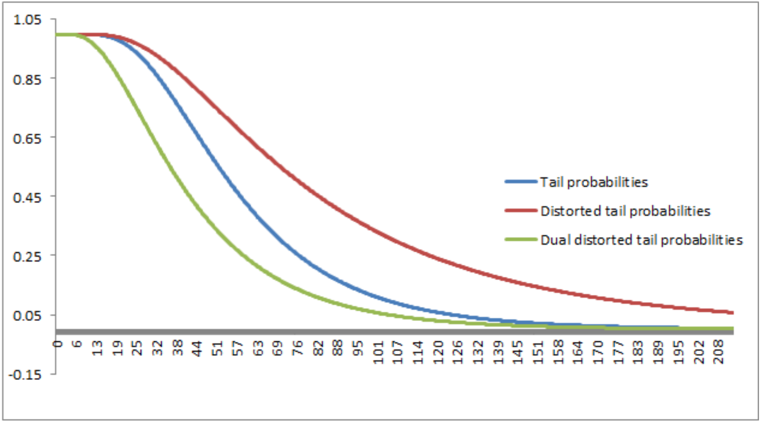
$$\begin{aligned} g \text{ is r.c.} \quad &\Leftrightarrow \quad \overline{g} \text{ is l.c.} \\ g \text{ is l.c.} \quad &\Leftrightarrow \quad \overline{g} \text{ is r.c.} \end{aligned}$$

- convex and concave functions

$$\begin{aligned} g \text{ is convex} \quad &\Leftrightarrow \quad \overline{g} \text{ is concave} \\ g \text{ is concave} \quad &\Leftrightarrow \quad \overline{g} \text{ is convex} \end{aligned}$$

3 – Dual distortion function

Example: concave distortion function



Exercise

- Consider a r.v. X with cdf F_X.
- The distortion function g is defined as

$$g(q) = \mathbb{I}(q > 1 - p).$$

- Determine the dual distortion function $\overline{g}$.
- Compare the distorted expectations $\rho_g[X]$ and $\rho_{\overline{g}}[X]$.

Theorem

Theorem

For any r.v. X, we have that

$$\rho_{\overline{g}}[X] = -\rho_g[-X],$$

and also

$$\rho_g[X] = -\rho_{\overline{g}}[-X].$$

- Exercise: Prove this theorem.

Comparing distorted expectations

- The distorted expectation hypothesis:

$$\text{Prefer loss } X \text{ over } Y \Leftrightarrow \rho_g[w - X] \geq \rho_g[w - Y].$$

  - A decision maker values wealth levels by using a distortion function g.
  - The decision maker is said to be a *distorted expectation maximizer*.
- Preferences are independent of initial wealth:

$$\rho_g[w - X] = w + \rho_g[-X].$$

- Preferences are invariant up to positive linear transformations.
- Dual theory:
  - instead of using *utility functions*, we now use *distorted expectations*.
  - In the dual theory, we compare *monetary units*, while in expected utility theory we compare *utility levels*.

The independence axiom

- Axiomatic framework: Yaari (1987)
  - Any decision maker whose behavior is in accordance with a given set of 'rational' axioms, is a **distorted expectation maximizer**.
  - The set of axioms is the same as in expected utility theory, except for the independence axiom.
- For any random losses X, Y and Z, their comonotonic modification X^c, Y^c and Z^c and $p \in [0,1]$, one has that

$$\begin{aligned} & \text{Prefer loss } X \text{ over loss } Y \\ \Rightarrow \quad & \text{Prefer loss } pX^c + (1-p)Z^c \text{ over loss } pY^c + (1-p)Z^c \end{aligned}$$

- Interpretation: Adding the loss Z^c to your portfolio cannot serve as a **hedge** for the losses X^c and Y^c.

Distorted expectations and utility levels

- Consider a decision maker with initial wealth w, facing a loss X.
- Expected terminal wealth level:

$$\mathbb{E}[w-X] = \int_0^1 F^{-1}_{w-X}(1-q)\mathrm{d}q.$$

- Expected utility level of terminal wealth (u is r.c.):

$$\mathbb{E}[u(w-X)] = \int_0^1 u\left(F^{-1}_{w-X}(1-q)\right)\mathrm{d}q.$$

- Distorted expectation of terminal wealth (g is l.c.):

$$\rho_g[w-X] = \int_0^1 F^{-1}_{w-X}(1-q)\mathrm{d}g(q).$$

Risk averse decision makers

- Definition:
  - A decision maker is *risk averse* if he/she has a convex distortion function.
- If g is a convex distortion function:

$$g\left(\overline{F}_X(x)\right) \leq \overline{F}_X(x).$$

  - The tail probabilities related to random levels of wealth are *underestimated*.
- Attitude towards risk:
  - Prefer certainty over uncertainty:

$$\rho_g[w - X] \leq \rho_g[w - \mathbb{E}[X]].$$

- Attitude towards wealth:
  - The satisfaction of gaining an additional Euro is independent of the initial wealth level.

Distorted expectation theory and insurance

- Risk averse individual:
  - facing a loss $X \geq 0$,
  - distortion function g,
  - initial wealth w.
- Risk averse insurer:
  - accepts X for a premium P,
  - distortion function G,
  - initial wealth W.
- Under what conditions is an insurance contract feasible?
  - From the viewpoint of the individual,
  - from the viewpoint of the insurer.

Distorted expectation theory and insurance

- Risk averse individual:
  - facing a loss $X \geq 0$,
  - distortion function g,
  - initial wealth w.
- Risk averse insurer:
  - accepts X for a premium P,
  - distortion function G,
  - initial wealth W.
- Under what conditions is an insurance contract feasible?
  - From the viewpoint of the individual,
  - from the viewpoint of the insurer.

Viewpoint of the insured

- Consider a decision maker with distortion function g, having initial wealth w and facing a loss X. He can buy insurance for a premium P.
  - He is only willing to underwrite the insurance if

$$\rho_g[w-P] \geq \rho_g[w-X].$$

- Maximal premium P^M he is willing to pay follows from:

$$\rho_g[w-P^M] = \rho_g[w-X].$$

- Solution:

$$P^M = -\rho_g[-X] = \rho_{\overline{g}}[X].$$

- Risk aversion leads to:

$$P^M \geq \mathbb{E}[X].$$

Viewpoint of the insurer

- Consider an insurer with distortion function G, having initial wealth W.
- The insurer is willing to insure a loss X at a premium P if

$$\rho_G[W] \leq \rho_G[W + P - X].$$

- Minimal premium P^m he requires follows from:

$$\rho_G[W] = \rho_G[W + P^m - X].$$

- Solution:

$$P^m = -\rho_G[-X] = \rho_{\overline{G}}[X].$$

- Risk aversion leads to

$$P^m \geq \mathbb{E}[X].$$

- The contract is feasible if $P^m \leq P \leq P^M$.

Theorem (Additivity for comonotonic risks)

If g is a distortion function and $(X_1^c, X_2^c, \dots, X_n^c)$ is a comonotonic modification of $(X_1, X_2, \dots, X_n)$, then

$$\rho_g[S^c] = \sum_{i=1}^{n} \rho_g[X_i].$$

Theorem

If g is a distortion function and (X_1^c, X_2^c) is a comonotonic modification of (X_1, X_2), then

$$\rho_g[w - X_1^c - X_2^c] = \rho_g[w - X_1^c] + \rho_g[w - X_2^c] - w.$$