Chapter 5: Yaari’s Dual Theory

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1. Distorted expectations

2. Distorted expectations

3. Discrete and continuous distortion functions

4. Dual theory of choice under risk
   Insurance
1 – Introduction

The problem of choosing between risks

- Setting: A decision maker can choose between two (random) losses: \( X \) and \( Y \).

- Question: What is the ‘optimal’ choice?

- Naive approach: Compare the expectations:

  \[
  \text{Prefer loss } X \text{ over } Y \iff \mathbb{E}[X] \leq \mathbb{E}[Y].
  \]
The problem of choosing between risks

- **Remark 1:** What if $E[X] = E[Y]$?
  - Is the decision maker indifferent?
  - even if $X$ and $Y$ are behaving very differently?

- **Remark 2:** The devil is in the tails!
  - Decision makers are assumed to be risk-averse.
  - The main concern is to *avoid (extreme) losses.*
The problem of choosing between risks

The most important insight from Chapter 1:
- In order to understand which losses are preferred,
- we have to take into account the risk-preferences of the decision maker.

**Consequence 1:** Different decision makers make different choices.
- Risk-preferences differ between decision makers.

**Consequence 2:** We need to model qualitative preferences using quantitative models.
- For example: in chapter 1 we used utility functions are used to model risk-preferences,
- and expected utilities to order random losses.
Yaari’s theory of choice under risk is an alternative for the expected utility theory.

- Risk preferences of decision makers are captured in a distortion function $g$.
- A decision maker always maximizes his distorted wealth level.

Risk preferences: Distortion functions

- Instead of utility functions.
- We keep the outcomes of the losses, but consider subjective probabilities.

Risk ordering: Distorted wealth levels

- Instead of expected utility.
- In Yaari’s theory, we compare expectations, but under a distorted probability measure.
Distortion risk measures:
- Generate coherent risk measures: See later chapter.

Sublinear expectations:
- A distorted expectation is a generalization of the classical expectation.
- It is an expectation under distorted probability levels.
- Under some continuity conditions, a distorted expectation is a classical expectation under a new probability measure.

Bid-ask pricing:
- A fundamental approach to justify bid-ask prices in asset pricing.
1 – Distorted expectations

Introduction

- Consider the gains \( X \) and \( Y \):
  
  \[ P[X = 1] = 1 \quad \text{and} \quad P[Y = x] = \begin{cases} 
  0.01, & x = 0; \\
  0.89, & x = 1; \\
  0.1, & x = 5.
\end{cases} \]

- Consider the gains \( V \) and \( W \):
  
  \[ P[V = x] = \begin{cases} 
  0.89, & x = 0; \\
  0.11, & x = 1.
\end{cases} \quad \text{and} \quad P[W = x] = \begin{cases} 
  0.9, & x = 0; \\
  0.1, & x = 5.
\end{cases} \]

- Empirical studies reveal that many people prefer \( X \) over \( Y \), but \( W \) over \( V \).
Expected utility maximizers who prefer $X$ over $Y$, will also prefer $V$ over $W$.

However, if a decision maker prefers $X$ over $Y$, but $W$ over $V$, then he/she can never be expected utility maximizers.

Paradox in expected utility theory.
A distortion function is a non-decreasing function $g : [0, 1] \rightarrow [0, 1]$ such that $g(0) = 0$ and $g(1) = 1$.

- $g$ is non-decreasing implies that $g$ is continuous and differentiable on $[0, 1]$, almost everywhere.
- A distortion function is said to be concave (convex) if it is concave (convex) on $[0, 1]$ without jumps in 0 and 1.
- Concave and convex distortion functions are continuous on $(0, 1)$.
Consider a random variable $X$. The tail probability $\bar{F}_X(x)$ is given by

$$\bar{F}_X(x) = P [X > x].$$

The function $g$ is used to distort the (tail) probabilities:

- If $g$ is concave: $g (\bar{F}_X(x)) \geq \bar{F}_X(x)$.
- If $g$ is convex: $g (\bar{F}_X(x)) \leq \bar{F}_X(x)$.
- **Exercise**: Prove these inequalities.
Consider a random variable $X$:

$$X \overset{d}{=} LN(\mu, \sigma^2).$$

Tail probabilities:

$$F_X(x) = \mathbb{P}[X > x] = \Phi \left( \frac{\mu - \ln x}{\sigma} \right),$$  

where $\Phi$ is the cdf of a standard normal distribution.

▶ Exercise: Prove this equality.

Distorted tail probabilities:

▶ $g(x) = \sqrt{x}$.

$$g(F_X(x)) = \sqrt{\Phi \left( \frac{\mu - \ln x}{\sigma} \right)}.$$
Distorted expectations

Distortion function: Example

- Parameters: $\mu = 4$ and $\sigma = 0.5$.
- Mean: $E[X] = e^{\mu + \frac{\sigma^2}{2}} \approx 62$. 
Consider a distortion function $g$. The distorted expectation of the r.v. $X$, notation $\rho_g[X]$, is

$$\rho_g[X] = -\int_{-\infty}^{0} [1 - g(F_X(x))] \, dx + \int_{0}^{+\infty} g(F_X(x)) \, dx,$$

provided both integrals are finite.

- The functional $\rho_g$ is called the \textit{distortion risk measure} (with distortion function $g$).
- Both integrals are assumed to be finite, which implies that $\rho_g[X]$ is finite.
The expectation of $X$:

$$E[X] = -\int_{-\infty}^{0} (1 - F_X(x)) \, dx + \int_{0}^{+\infty} F_X(x) \, dx.$$ 

Distorted expectation of $X$:

$$\rho_g[X] = -\int_{-\infty}^{0} \left[1 - g \left(F_X(x)\right)\right] \, dx + \int_{0}^{+\infty} g \left(F_X(x)\right) \, dx.$$ 

Interpretation:

- If $g$ is left continuous, then $g \left(F(x)\right)$ is non-increasing and right continuous with values in $[0, 1]$.

- The distorted expectation $\rho_g[X]$ can be interpreted as an expectation of $X$, but the tail probabilities $F_X(x)$ are replaced by the distorted tail probabilities $g \left(F_X(x)\right)$. 
2 – Distorted expectations

Graphical interpretation for concave \( g \)

\[
E[X] = I - (II + II') \quad \rho_g[X] = (I + I') - II \geq E[X]
\]
A distortion function is **continuous** if it is continuous on \([0, 1]\).

- **Example**
  - Distortion function
    
    \[ g(q) = q, \quad \text{for} \quad q \in [0, 1]. \]

  - Corresponding distorted expectation
    
    \[ \rho_g[X] = \mathbb{E}[X]. \]

  - The distortion function \( g \) doesn’t distort the probabilities.

**Concave/convex functions**

- If \( g \) is concave: \( \rho_g[X] \geq \mathbb{E}[X]. \)

- If \( g \) is convex: \( \rho_g[X] \leq \mathbb{E}[X]. \)

- **Exercise**: Prove these inequalities.
Theorem

For any distortion function $g$ and any r.v.’s $X$ and $Y$, the following properties hold:

1. **Positive homogeneity:**
   
   $$ \rho_g [aX] = a \rho_g [X], \quad a > 0; $$

2. **Translation invariance:**
   
   $$ \rho_g [X + b] = \rho_g [X] + b, \quad b \in \mathbb{R}; $$

3. **Preserving stochastic dominance:**
   
   if $F_X(x) \geq F_Y(x)$, for all $x \in \mathbb{R} \Rightarrow \rho_g [X] \leq \rho_g [Y].$
Consider the following r.c. distortion function
\[ g(q) = \mathbb{I}(q \geq 1 - p), \quad 0 \leq q \leq 1. \]

Corresponding distorted expectation:
\[ \rho_g[X] = F_X^{-1+}(p). \]

The condition that \( g \) is r.c. is essential:
- Consider the l.c. distortion function
  \[ g(q) = \mathbb{I}(q > 1 - p), \quad 0 \leq q \leq 1. \]
- Corresponding distorted expectation:
  \[ \rho_g[X] = F_X^{-1}(p). \]
Dual distortion function

Definition

The dual distortion function $\bar{g}$ of the distortion function $g$ is

$$\bar{g}(q) = 1 - g(1 - q).$$

- $\bar{g}$ is again a distortion function.
- Right and left continuous functions

$$g \text{ is r.c. } \iff \bar{g} \text{ is l.c.}$$

$$g \text{ is l.c. } \iff \bar{g} \text{ is r.c.}$$

- Convex and concave functions

$$g \text{ is convex } \iff \bar{g} \text{ is concave}$$

$$g \text{ is concave } \iff \bar{g} \text{ is convex}$$
3 – Dual distortion function

Example: concave distortion function
Consider a r.v. $X$ with cdf $F_X$.

The distortion function $g$ is defined as

$$g(q) = \mathbb{1}(q > 1 - p).$$

Determine the dual distortion function $\bar{g}$.

Compare the distorted expectations $\rho_g[X]$ and $\rho_{\bar{g}}[X]$. 
### Theorem

For any r.v. $X$, we have that

$$\rho_g[X] = -\rho_g[-X],$$

and also

$$\rho_g[X] = -\rho_g[-X].$$

**Exercise:** Prove this theorem.
Comparing distorted expectations

- The distorted expectation hypothesis:

\[
\text{Prefer loss } X \text{ over } Y \Leftrightarrow \rho_g[w - X] \geq \rho_g[w - Y].
\]

- A decision maker values wealth levels by using a distortion function \( g \).
- The decision maker is said to be a distorted expectation maximizer.

- Preferences are independent of initial wealth:

\[
\rho_g[w - X] = w + \rho_g[-X].
\]

- Preferences are invariant up to positive linear transformations.

Dual theory:

- Instead of using utility functions, we now use distorted expectations.
- In the dual theory, we compare monetary units, while in expected utility theory we compare utility levels.
The independence axiom

- **Axiomatic framework: Yaari (1987)**
  - Any decision maker whose behavior is in accordance with a given set of ‘rational’ axioms, is a **distorted expectation maximizer**.
  - The set of axioms is the same as in expected utility theory, except for the independence axiom.

- For any random losses $X$, $Y$ and $Z$, their comonotonic modification $X^c$, $Y^c$ and $Z^c$ and $p \in [0, 1]$, one has that

  Prefer loss $X$ over loss $Y$
  \[ \Rightarrow \text{ Prefer loss } pX^c + (1 - p)Z^c \text{ over loss } pY^c + (1 - p)Z^c \]

- **Interpretation:** Adding the loss $Z^c$ to your portfolio cannot serve as a **hedge** for the losses $X^c$ and $Y^c$. 
Consider a decision maker with initial wealth $w$, facing a loss $X$.

**Expected terminal wealth level:**

$$
\mathbb{E}[w - X] = \int_0^1 F_{w-X}^{-1} (1 - q) dq.
$$

**Expected utility level of terminal wealth ($u$ is r.c.):**

$$
\mathbb{E}[u(w - X)] = \int_0^1 u \left( F_{w-X}^{-1} (1 - q) \right) dq.
$$

**Distorted expectation of terminal wealth ($g$ is l.c.):**

$$
\rho_g[w - X] = \int_0^1 F_{w-X}^{-1} (1 - q) dg(q).
$$
Risk averse decision makers

- **Definition:**
  - A decision maker is *risk averse* if he/she has a convex distortion function.
  
- If \( g \) is a convex distortion function:
  \[ g \left( F_X(x) \right) \leq F_X(x). \]

- The tail probabilities related to random levels of wealth are *underestimated*.

- **Attitude towards risk:**
  - Prefer certainty over uncertainty:
    \[ \rho_g[w - X] \leq \rho_g[w - \mathbb{E}[X]]. \]

- **Attitude towards wealth:**
  - The satisfaction of gaining an additional Euro is *independent* of the initial wealth level.
Distorted expectation theory and insurance

- **Risk averse individual:**
  - facing a loss $X \geq 0$,
  - distortion function $g$,
  - initial wealth $w$.

- **Risk averse insurer:**
  - accepts $X$ for a premium $P$,
  - distortion function $G$,
  - initial wealth $W$.

**Under what conditions is an insurance contract feasible?**
- From the viewpoint of the individual,
- from the viewpoint of the insurer.
Risk averse individual:
- facing a loss $X \geq 0$,
- distortion function $g$,
- initial wealth $w$.

Risk averse insurer:
- accepts $X$ for a premium $P$,
- distortion function $G$,
- initial wealth $W$.

Under what conditions is an insurance contract feasible?
- From the viewpoint of the individual,
- from the viewpoint of the insurer.
Consider a decision maker with distortion function $g$, having initial wealth $w$ and facing a loss $X$. He can buy insurance for a premium $P$. He is only willing to underwrite the insurance if

$$\rho_g[w - P] \geq \rho_g[w - X].$$

Maximal premium $P^M$ he is willing to pay follows from:

$$\rho_g[w - P^M] = \rho_g[w - X].$$

Solution:

$$P^M = -\rho_g[-X] = \rho_{\bar{g}}[X].$$

Risk aversion leads to:

$$P^M \geq \mathbb{E}[X].$$
Consider an insurer with distortion function $G$, having initial wealth $W$.

The insurer is willing to insure a loss $X$ at a premium $P$ if

$$\rho_G[W] \leq \rho_G[W + P - X].$$

Minimal premium $P^m$ he requires follows from:

$$\rho_G[W] = \rho_G[W + P^m - X].$$

Solution:

$$P^m = -\rho_G[-X] = \rho_G[X].$$

Risk aversion leads to

$$P^m \geq E[X].$$

The contract is feasible if $P^m \leq P \leq P^M$. 
Theorem (Additivity for comonotonic risks)

If $g$ is a distortion function and $(X_1^c, X_2^c, \ldots, X_n^c)$ is a comonotonic modification of $(X_1, X_2, \ldots, X_n)$, then

$$
\rho_g[S^c] = \sum_{i=1}^{n} \rho_g[X_i].
$$

Theorem

If $g$ is a distortion function and $(X_1^c, X_2^c)$ is a comonotonic modification of $(X_1, X_2)$, then

$$
\rho_g[w - X_1^c - X_2^c] = \rho_g[w - X_1^c] + \rho_g[w - X_2^c] - w.
$$