KU LEUVEN

Foundations of Quantitative Risk Measurement

Chapter 6: Risk measures

Jan Dhaene and Daniël Linders

November, 2019

1. Introduction

2. Risk measures
   - Definition
   - Coherent risk measures

3. Value-at-Risk and Tail Value-at-Risk

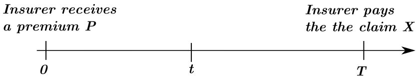
4. The required solvency capital

5. The Haezendonck-Goovaerts risk measure

6. VaR, TVaR and comonotonicity

1 – The insurance business

- Inverted production cycle:
  - Insured: pays a fixed premium.
  - Insurer: accept the risk to pay the claim amounts related to possible future events.
  - Similar for insurer/reinsurer or reinsurer/reinsurer.
- Solvency:
  - The insurer has to set a premium, such that he is able to pay all the future benefits to the policy holders.
  - When a claim has to be paid, the insurer should be solvent.
- Example:
  - X and/or T can be random.



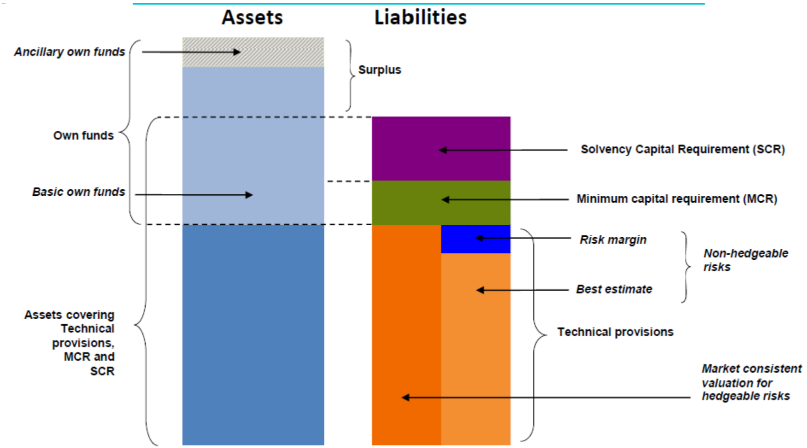
- Underwriting risk:
  - The risk that the premiums are not sufficient to cover the obligations of the insurer.
  - Underwriting risk arises when the insurer underestimates the future losses.
- Credit risk:
  - The risk that a counter party will not meet his obligations.
  - The premiums are invested in bonds (and/or other assets), which can default.
- Market risk:
  - Exposure to the uncertain future market value of the investment portfolio.
- Other risks:
  - Liquidity risk, operational risk, ...

Technical provisions

- Technical provisions:
  - also called: actuarial reserves or best estimate;
  - the amount the insurer has to hold in order to cover the **expected** future claims;
  - fair value of the future liabilities: the amount another party is willing to pay to take over the insurance business.
- Valuation = Market Consistent
  - Mark-to-market:
    - hedging the liabilities using a replicating portfolio gives a unique price.
  - Mark-to-model
    - Model-based approach to determine the expectation of the future liabilities.
    - Model risk: use a prudence margin,

- What is solvency:
  - Solvency refers to the ability of the insurer to meet his obligations to pay the present and future claims related to policyholders.
- Solvency rules:
  - Imposed by the regulatory authority.
  - Protecting policyholders.
- Solvency Capital Requirements:
  - In addition to the technical provisions V, the insurer has to hold a capital buffer.
  - Minimal level of capital an insurer has to hold such that the insurer is *very likely* to be able to meet his future obligations.

http://www.solvency-2.com



Definition (Risk measure)

Consider a set Λ of real-valued r.v.'s. The function ρ assigning a real number $\rho[X]$ to a r.v. $X \in \Lambda$, is called a *risk measure*.

- ρ is also called a *risk measure with domain* Λ.
  - In most cases, Λ is not specified and has to be taken 'as broad as possible'.
- X represents a loss over a given reference period
  - $\rho[X]$ captures the 'risk' in a real number.
- Example:
  - Standard deviation risk measure
  - $\rho[X] =$ Expected value $+$ safety loading:

$$\rho[X] = \mu_X + \lambda\sigma_X, \quad \lambda \geq 0.$$

References

- Risk measures and premium principles:
  - Goovaerts, De Vylder & Haezendonck (1984), Kaas, Goovaerts, Dhaene & Denuit (2008).
- Risk measures and capital requirements:
  - Artzner (1999), Wirch & Hardy (2000).

Distributional properties that a risk measure may satisfy

- Law invariance:

$$X \overset{\mathsf{d}}{=} Y \Rightarrow \rho[X] = \rho[Y].$$

- Preserving stop-loss order:

$$X \preceq_{sl} Y \Rightarrow \rho[X] \leq \rho[Y].$$

- Comonotonic additivity:
  - For any comonotonic r.v.'s X, Y:

$$\rho[X+Y] = \rho[X] + \rho[Y].$$

- Stop-loss premium with retention $K \in \mathbb{R}$:
  - The risk measure ρ is defined as

$$\rho\left[X\right] = \mathbb{E}\left[\left(X - K\right)_{+}\right].$$

- Exercise
  - Prove that ρ is law invariant.
  - Prove that ρ preserves stochastic dominance and stop-loss order.
- Prove also that ρ is not comonotonic additive.

2 – Properties

Distribution-free properties that a risk measure may satisfy

- Monotonicity:

$$X \leq Y \Rightarrow \rho[X] \leq \rho[Y].$$

- Positive homogeneity:
  - For any $a > 0$,

$$\rho[aX] = a\rho[X].$$

- Translation invariance:
  - For any $b \in \mathbb{R}$

$$\rho[X + b] = \rho[X] + b.$$

- Subadditivity:

$$\rho[X + Y] \leq \rho[X] + \rho[Y].$$

Definition (Coherent risk measure)

A risk measure is said to be coherent if it satisfies the properties

1. monotonicity;
2. positive homogeneity;
3. translation invariance;
4. subadditivity.

[1] Artzner et al. (1999)

Example of a coherent risk measure

- Upper expectation[2]
  - Π is a subset of all probability measures on the measurable space $(\Omega, \mathcal{F})$

$$\rho_{\Pi}[X] = \sup \{\mathbb{E}_{\mathbb{P}}[X] \mid \mathbb{P} \in \Pi\}.$$

  - Exercise: Prove that ρ_{Π} is a coherent risk measure.
- Interpretation:
  - The elements of Π are *generalized scenarios*.
  - Expectation of X with respect to a *worst-case scenario*.
- Upper expectations can also be considered as a $\mathcal{G}$-expectations.[3]

[2] see Huber (1981)
[3] see e.g. Peng (2007)

Law invariance and upper expectation

- Define the risk measure ρ as:

$$\rho[X] = \max\{\mathbb{E}[X], \mathbb{E}[X \mid Z > z]\},$$

  with $\mathbb{P}[Z \leq z] > 0$.

- ρ is an upper expectation:
  - $\mathbb{P}$ is the real-world probability measure.
  - $\mathbb{Q}$ is a distorted probability measure:

$$\mathbb{Q}[A] = \mathbb{P}[A \mid Z > z], \text{ for an event } A.$$

  - Then:

$$\rho[X] = \max\{\mathbb{E}_{\mathbb{P}}[X], \mathbb{E}_{\mathbb{Q}}[X]\}.$$

- ρ is not law invariant:
  - For $Y \overset{d}{=} Z$ and Y and Z independent: $\rho[Y] \neq \rho[Z]$.

Risk and uncertainty

- In Knight (1921), Frank Knight makes a distinction between *risk* and *uncertainty*.
- *Risk* refers to the uncertainty about the realization of a loss X.
  - Risk is modeled by the cdf F_X of X, which gives the probability of X being in a given region.
  - If the cdf F_X is known, the expectation $\mathbb{E}[X]$ can be determined.
- In practical situations, the cdf F_X is not fully known.
  - We only have partial information: $F_X \in \mathcal{P}$.
  - This induces an extra source of *uncertainty*.
- A robust estimate for the expectation is the *upper expectation*

$$\sup \left\{ \mathbb{E}_F \left[X \right] \mid F \in \mathcal{P} \right\}.$$

Theorem

Consider a risk measure ρ. The following statements are equivalent:

1. *The risk measure ρ is coherent.*
2. *There exist a set Π of probability measures such that*

$$\rho\left[X\right] = \sup\left\{\mathbb{E}_{\mathbb{P}}\left[X\right] \mid \mathbb{P} \in \Pi\right\}.$$

- A proof for the case where Ω is finite: Huber (1981).
- The more general case is proven in Delbaen (2002).

Definition

For any p in $(0,1)$ the Value-at-Risk at level p is defined by

$$VaR_p[X] = \inf\{x \in \mathbb{R} \mid F_X(x) \geq p\}.$$

- Value-at-Risk = Quantile
  - $\mathsf{VaR}_p[X] = F_X^{-1}(p)$
- Interpretation:
  - Take p close to 1.
  - Probability that X exceeds the threshold is small:

$$\mathbb{P}\left[X > \mathsf{VaR}_p[X]\right] = 1 - \underbrace{F_X\left(\mathsf{VaR}_p[X]\right)}_{\geq p} \leq 1 - p.$$

- Value-at-Risk is increasing in p
  - $p < q \Rightarrow \mathsf{VaR}_p[X] \leq \mathsf{VaR}_q[X]$.
- Positive homogeneous and translation invariant:
  - For $a > 0$ and $b \in \mathbb{R}$: $\mathsf{VaR}_p[aX + b] = a\mathsf{VaR}_p[X] + b$.
  - Exercise: Prove these properties.
- VaR is not a coherent risk measure
  - Exercise: give a counterexample.

The VaR for avoiding bankruptcy

- Solvency capital requirement:
  - X = future loss/liabilities.
  - V = technical provision (fair valuation of X).
  - $K[X]$ = solvency capital

$$K[X] = \text{VaR}_p[X] - V$$

  - Extra buffer of capital for bad times[4].
- Interpretation:
  - $K[X] = \text{VaR}_p[X - V]$.
  - $X - V$ = loss which is not covered by the technical provisions

$$\mathbb{P}[X - V > K[X]] = \mathbb{P}[X > K[X] + V] \leq 1 - p.$$

  - Insolvency = technical provision + capital buffer are not sufficient to cover to loss:
    - ★ For p close to 1, the probability that the insurer is insolvent is small.

[4] p should be sufficiently big, such that $K[X] > 0$.

VaR for normal and lognormal distributions

- The cdf of a standard normal distribution $= \Phi$.
- Assume $Y \overset{\mathsf{d}}{=} N\left(\mu, \sigma^2\right)$. Then

$$\mathsf{VaR}_p\left[Y\right] = \mu + \sigma \Phi^{-1}\left(p\right).$$

- For a lognormal r.v. $X \overset{\mathsf{d}}{=} \mathsf{e}^{Y}$:

$$\mathsf{VaR}_p\left[X\right] = \mathsf{e}^{\mu + \sigma \Phi^{-1}(p)}.$$

VaR and low frequency - high impact events

- Consider the r.v. X:

$$\mathbb{P}[X = -10000] = 0.999 \quad \text{and} \quad \mathbb{P}[X = 10\,000\,000] = 0.001.$$

- high probability of having a gain, but a low probability of having a big loss.
- The VaR does not detect there is an extreme risk:

$$\text{VaR}_{0.95}[X] = -10000.$$

- However, if something goes wrong, i.e. if the loss exceeds the VaR, the loss is enormous:

$$\mathbb{E}[X \mid X > \text{VaR}_{0.95}[X]] = 10\,000\,000.$$

- *"I believe that VaR is the alibi that bankers will give shareholders (and the bailing-out taxpayer) to show documented due diligence, and will express that their blow-up came from truly unforeseeable circumstances and events with low probability not from taking large risks that they didn't understand. I maintain that VaR encourages untrained people to take misdirected risks with shareholders', and ultimately the taxpayers', money."*

  ——– Nassim Taleb 1997.

- *"The risk-taking model that emboldened Wall Street to trade with impunity is broken and everyone is coming to the realization that no algorithm can substitute for old-fashioned due diligence. VaR failed to detect the scope of the market's collapse. The past months have exposed the flaws of a financial measure based on historical prices."*

  ——– Financial Reporter Christine Harper, January 2008.

Definition

For any p in $(0,1)$ the Tail Value-at-Risk at level p is defined by

$$TVaR_p[X] = \frac{1}{1-p}\int_p^1 VaR_q[X]\,dq.$$

- Interpretation:
  - TVaR is an average of VaR's.
  - Exercise: prove that $\text{VaR}_p[X] \leq \text{TVaR}_p[X]$.
- What is bad?
  - Probability that X exceeds $\text{VaR}_p[X]$ is small, but not 0.
  - If X exceeds $\text{VaR}_p[X]$, what will be the loss?
  - Bad times are those when $X \in [\text{VaR}_p[X], \text{TVaR}_p[X]]$.

TVaR as a cushion for extreme losses

- Consider the loss X :

$$\begin{aligned} \mathbb{P}\left[X=0\right] &= 0.6, \\ \mathbb{P}\left[X=100\right] &= 0.37, \\ \mathbb{P}\left[X=10000\right] &= 0.02, \\ \mathbb{P}\left[X=100000\right] &= 0.01. \end{aligned}$$

- $\text{VaR}_{0.95}\left[X\right]=100.$
  - Probabillity of a loss bigger than 100 is ‘only’ 5%.
  - What can we expect if the loss **will** exceed $\text{VaR}_{0.95}\left[X\right]$?
- $\text{TVaR}_{0.95}\left[X\right]=24040.$
  - If the loss will exceed $\text{VaR}_{0.95}\left[X\right]$, then it will be, on average, equal to $\text{TVaR}_{0.95}\left[X\right]$.

Properties of TVaR

- Tail Value-at-Risk is increasing in p
  - $p < q \Rightarrow \mathsf{TVaR}_p[X] \leq \mathsf{TVaR}_q[X]$.
- Positive homogeneous and translation invariant:
  - For $a > 0$ and $b \in \mathbb{R}$: $\mathsf{TVaR}_p[aX + b] = aT\mathsf{VaR}_p[X] + b$.
  - Exercise: Prove these properties.
- TVaR is a coherent risk measure.
  - We give a prove in the next chapter.

- Solvency capital requirement:
  - X is a future loss/liability.
  - V = technical provision
  - $K[X]$ = solvency capital

$$K[X] = \text{TVaR}_p[X] - V.$$

- Interpretation:
  - $K[X] = \text{TVaR}_p[X - V] \geq \text{VaR}_p[X - V]$.
  - $X - V$ = loss which is not covered by the technical provisions

$$\mathbb{P}\left[X - V > \text{VaR}_p[X - V]\right] \leq 1 - p.$$

  - First buffer = $\text{VaR}_p[X - V]$.
  - Second buffer = $\text{TVaR}_p[X - V]$.

- Conditional Tail Expectation:
  - For any $p \in (0,1)$ and any r.v. X, the Conditional Tail Expectation at level p is
  $$\mathsf{CTE}_p[X] = \mathbb{E}\left[X \mid X > \mathsf{VaR}_p[X]\right].$$
  - $\mathsf{VaR}_p[X] \leq \mathsf{CTE}_p[X]$.
- Expected Shortfall:
  - For any $p \in (0,1)$ and any r.v. X, the Expected Shortfall at level p is
  $$\mathsf{ESF}_p[X] = \mathbb{E}\left[\left(X - \mathsf{VaR}_p[X]\right)_+\right].$$
  - Solvency capital $=$ Value-at-Risk $+$ Expected Shortfall.

Theorem

For $p \in (0,1)$, we have that

$$TVaR_p[X] = F_X^{-1}(p) + \frac{1}{1-p} ESF_p[X],$$

$$CTE_p[X] = F_X^{-1}(p) + \frac{1}{1 - F_X\left[F_X^{-1}(p)\right]} ESF_p[X],$$

$$CTE_p[X] = TVaR_{F_X\left[F_X^{-1}(p)\right]}[X].$$

- Remarks:
  - In general, $F_X\left[F_X^{-1}(p)\right]$ is not always equal to p.
  - If F_X is a continuous distribution function:

$$\text{CTE}_p[X] = \text{TVaR}_p[X].$$

- X is normal distributed with mean μ_X and variance σ_X^2.
- Value-at-Risk:
  - For $p \in (0,1)$

$$\mathsf{VaR}_p\left[X\right] = \mu_X + \sigma_X \Phi^{-1}\left(p\right).$$

- Stop-loss premium:

$$\mathbb{E}\left[\left(X-K\right)_+\right] = \sigma_X \phi\left(\frac{K-\mu_X}{\sigma_X}\right) - \left(K-\mu_X\right)\left[1-\Phi\left(\frac{K-\mu_X}{\sigma_X}\right)\right].$$

- Expected Shortfall:

$$\begin{aligned}\mathsf{ESF}_p\left[X\right] &= \mathbb{E}\left[\left(X-\mathsf{VaR}_p\left[X\right]\right)_+\right] \\ &= \sigma_X \phi\left(\Phi^{-1}\left(p\right)\right) - \sigma_X \Phi^{-1}\left(p\right)\left(1-p\right).\end{aligned}$$

- Conditional Tail Expectation:
  - For $p \in (0,1)$

$$
\begin{aligned}
\mathsf{CTE}_p[X] &= F_X^{-1}(p) + \frac{1}{1 - F_X\left[F_X^{-1}(p)\right]} \mathsf{ESF}_p[X] \\
&= \mu_X + \sigma_X \frac{\phi\left(\Phi^{-1}(p)\right)}{1-p}.
\end{aligned}
$$

  - $\mathsf{TVaR}_p[X] = \mathsf{CTE}_p[X]$.

- Consider the r.v. X with distribution function

$$F_X(x) = \begin{cases} x & \text{, if } 0 \leq x < 0.85, \\ 0.85 & \text{, if } 0.85 \leq x < 0.9, \\ 0.95 & \text{, if } 0.9 \leq x < 0.95 \\ x & \text{, if } 0.95 \leq x \leq 1. \end{cases}$$

- Value-at-Risk:
  - Prove that the value-at-risk is given by:

$$\text{Var}_p[X] = \begin{cases} p & \text{, if } 0 < p \leq 0.85, \\ 0.9 & \text{, if } 0.85 < p \leq 0.95 \\ p & \text{, if } 0.95 < p \leq 1. \end{cases} .$$

- Conditional Tail Expectation:
  - Take $p = 0.9$. Prove that

$$\text{CTE}_p[X] = 0.975.$$

- Regulators view:
  - V = technical provisions of a future loss X.
  - Solvency capital: $\rho[X] - V$.
  - φ : risk measure to calculate the shortfall risk.
  - Cost of insolvency:
$$\varphi\left[(X - \rho[X])_+\right]$$
- Investors view:
  - Holding an extra amount of capital requires a return equal to ε :
$$(\rho[X] - V)\,\varepsilon.$$
- Cost function:
  - For $\varepsilon \in (0,1)$:
$$C(X, \rho[X]) = \varphi\left[(X - \rho[X])_+\right] + (\rho[X] - V)\,\varepsilon.$$
  - Find the optimal capital requirement K such that the cost function $C(X, K)$ is minimal.

Theorem

Take $0 < \varepsilon < 1$. *The cost function* $C(x, \rho[X])$ *defined by*

$$C(X, \rho[X]) = \mathbb{E}\left[(X - \rho[X])_+\right] + (\rho[X] - V)\,\varepsilon,$$

reaches its minumum if

$$\rho[X] = VaR_{1-\varepsilon}[X],$$

- Minimal value of the cost function:

$$C\left(X, \mathrm{VaR}_{1-\varepsilon}\left[X\right]\right) = \varepsilon\left(\mathrm{TVaR}_{1-\varepsilon}\left[X\right] - V\right).$$

- Take $\varepsilon = 1 - p$. For any K :

$$C\left(X, K\right) \geq \left(1 - p\right)\Big(\mathrm{TVaR}\left[X\right]_p - V\Big).$$

- So we find:

$$\mathrm{TVaR}_p\left[X\right] \leq K + \frac{1}{1-p}\mathbb{E}\left[\left(X - K\right)_+\right].$$

- Note that we have an equality if $K = \mathrm{VaR}_p[X]$.

Theorem

TVaR is a minimum:

$$\mathit{TVaR}_p[X] = \min_K \left\{ K + \frac{1}{1-p} \mathbb{E}\left[(X-K)_+ \right] \right\}.$$

- Consider an insurer, facing the loss $X \in (0, \max[X])$.
- Total available capital (including a solvency buffer)$= V$.
- Capital with extra risk buffer $= \rho$.
- The random variable Z is defined as:

$$Z = \frac{(X - V)_+}{\rho - V}.$$

  - If $X \leq V$, there is no extra capital needed and $Z = 0$.
  - If $Z \leq 1$, the extra buffer $\rho - V$ absorbs the unanticipated losses.
  - If $Z > 1$, the extra buffer $\rho - V$ is not sufficient to cover realized loss.

- We force $\mathbb{E}[Z]$ to be equal to $1-\alpha$, where $\alpha \in (0,1)$.
- The solution is then denoted by $\rho[X,V]$:

$$\mathbb{E}\left[\frac{(X-V)_+}{\rho[X,V]-V}\right] = 1-\alpha.$$

- We can then write:

$$\rho[X,V] = V + \frac{1}{1-\alpha}\mathbb{E}\left[(X-V)_+\right].$$

- We determine the capital level V, such that $\rho[X,V]$ is minimal.

- The **linear Haezendonck-Goovaerts risk measure** $\rho[X]$ is defined as:

$$\rho[X] = \inf_{V \in [0, \max[X]]} \rho[X, V].$$

- We can write:

$$\rho[X] = \inf_{V \in [0, \max[X]]} \left\{ V + \frac{1}{1-\alpha} \mathbb{E}\left[(X - V)_+ \right] \right\}$$

- We find that:

$$\rho[X] = TVaR_\alpha[X].$$

VaR, TVaR, ESF are additive for comonotonic risks

Theorem

Consider a comonotonic random vector $(X_1^c, X_2^c, \ldots, X_n^c)$ *and let* $S^c = \sum_{i=1}^n w_i X_i^c$*, where* w_i *are positive weigth factors. Then we have for all* $p \in (0,1)$ *that*

$$VaR_p\left[S^c\right] = \sum_{i=1}^n w_i VaR_p\left[X_i\right],$$

$$TVaR_p\left[S^c\right] = \sum_{i=1}^n w_i TVaR_p\left[X_i\right],$$

$$ESF_p\left[S^c\right] = \sum_{i=1}^n w_i ESF_p\left[X_i\right].$$

Exercise: prove this theorem.

- Continuous distribution functions:
  - Assume that the cdf of X_i^c are continuous.
  - Then we have that $CTE_p[X_i] = TVaR_p[X_i]$.
  - S^c has also a continuous distribution function: $CTE_p[S^c] = TVaR_p[S^c]$.
  - CTE is additive for comonotonic risks.
- In general: CTE is not additive for comonotonic risks
  - counter example: F_Y is uniform over $[0,1]$.
  - F_X is

$$F_X(x) = \begin{cases} x & \text{, if } 0 \leq x < 0.85, \\ 0.85 & \text{, if } 0.85 \leq x < 0.9, \\ 0.95 & \text{, if } 0.9 \leq x < 0.95 \\ x & \text{, if } 0.95 \leq x \leq 1. \end{cases}$$

  - For $p = 0.9$: $\text{CTE}_p[S^c] < \text{CTE}_p[X] + \text{CTE}_p[Y]$.