Foundations of Quantitative Risk Measurement

Subadditivity and risk aggregation

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0 – Outline

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   Convex upper bound
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Consider the risks $X_1, X_2, \ldots, X_n$.

The **aggregated risk** is denoted by $S$:

$$S = X_1 + X_2 + \ldots + X_n.$$ 

Knowledge of the cdf $F_S$ is required when determining risk measures for the aggregated risk $S$.

- The cdf $F_S$ is determined by
  - Marginal information: $F_{X_1}, F_{X_2}, \ldots, F_{X_n}$.
  - Dependence information: The copula $C$ connects the marginals.
1 – Risk aggregation

Introduction

- **Estimating the cdf $F_S$**
  - estimating the marginal cdfs is possible and can, often, be done accurately.
  - estimating the copula is harder, especially when $n$ is large.
- In general, the cdf $F_S$ will be unknown or too cumbersome to work with.
  - Determining $\text{TVaR}_p[S]$, $\text{VaR}_p[S]$, $\mathbb{E} \left[ (S - K)_+ \right]$ is not an easy task.
- In case we have **comonotonic risks**, it is possible to determine $\text{TVaR}_p[S^c]$, $\text{VaR}_p[S^c]$, $\mathbb{E} \left[ (S^c - K)_+ \right]$. 
Recall that TVaR can be written as:

$$\text{TVaR}_p [X] = F_X^{-1} (p) + \frac{1}{1 - p} \mathbb{E} \left[ (X - F_X^{-1}(p))_+ \right].$$

Take $p \in (0, 1)$. The stop-loss premium can be rewritten:

$$\mathbb{E} \left[ (X - F_X^{-1}(\alpha)(p))_+ \right] = \mathbb{E} \left[ (X - F_X^{-1}(p))_+ \right] - \left( F_X^{-1}(\alpha)(p) - F_X^{-1}(p) \right) (1 - p)$$

Combining the two equations results in

$$\text{TVaR}_p [X] = F_X^{-1}(\alpha)(p) + \frac{1}{1 - p} \mathbb{E} \left[ (X - F_X^{-1}(\alpha)(p))_+ \right].$$
Consider the comonotonic sum $S^c$

$$S^c = F_{X_1}^{-1}(U) + F_{X_2}^{-1}(U) + \ldots + F_{X_n}^{-1}(U).$$

TVaR is additive for comonotonic risks:

$$\text{TVaR}_p[S^c] = \text{TVaR}_p[X_1] + \text{TVaR}_p[X_2] + \ldots + \text{TVaR}_p[X_n].$$

Using the expression of TVaR in terms of alpha inverses, we find:

$$\mathbb{E} \left[ \left( S^c - F_{S^c}^{-1(\alpha)}(p) \right)_+ \right] = \sum_{i=1}^{n} \mathbb{E} \left[ \left( X_i - F_{X_i}^{-1(\alpha)}(p) \right)_+ \right].$$
Choose a \( K \in \left( F_{S^c}^{-1}(0), F_{S^c}^{-1}(1) \right) \). Then there always exists an \( \alpha_K \) such that
\[
K = F_{S^c}^{-1}(\alpha_K) (F_{S^c}(K)) .
\]

Take \( p = F_{S^c}(K) \), then we find
\[
\mathbb{E} \left[ (S^c - K)_+ \right] = \sum_{i=1}^{n} \mathbb{E} \left[ \left( X_i - F_{X_i}^{-1}(\alpha_K)(p) \right)_+ \right] .
\]

If \( K \neq \left( F_{S^c}^{-1}(0), F_{S^c}^{-1}(1) \right) \):
- \( K \leq F_{S^c}^{-1}(0) : \mathbb{E} \left[ (S^c - K)_+ \right] = \mathbb{E} [S] - K . \)
- \( K \geq F_{S^c}^{-1}(1) : \mathbb{E} \left[ (S^c - K)_+ \right] = 0 . \)
**Theorem**

For $K \in \left( F_{S^c}^{-1}(0), F_{S^c}^{-1}(1) \right)$, the stop-loss premiums $\mathbb{E} \left[ (S^c - K)_+ \right]$ can be written as

$$\mathbb{E} \left[ (S^c - K)_+ \right] = \sum_{i=1}^{n} \mathbb{E} \left[ (X_i - K_i)_+ \right],$$

where

$$K_i = F_{X_i}^{-1}(\alpha_K) \left( F_{S^c}(K) \right),$$

and $\alpha_K$ follows from

$$K = \sum_{i=1}^{n} K_i.$$

---

- We can prove a stronger result:

$$\left( \sum_{i=1}^{n} F_{X_i}^{-1}(U) - K \right)_+ = \sum_{i=1}^{n} \left( F_{X_i}^{-1}(U) - K_i \right)_+.$$
Assume that each cdf $F_{X_i}$ is strictly increasing.

Then, also $F_{S^c}$ is strictly increasing:

$$F_{S^c}^{-1}(\alpha_K) \left( F_{S^c}(K) \right) = F_{S^c}^{-1} \left( F_{S^c}(K) \right).$$

For $K \in \left( F_{S^c}^{-1}(0), F_{S^c}^{-1}(1) \right)$, we find

$$\mathbb{E} \left[ (S^c - K)_+ \right] = \sum_{i=1}^{n} \mathbb{E} \left[ (X_i - K_i)_+ \right],$$

where

$$K_i = F_{X_i}^{-1} \left( F_{S^c}(K) \right),$$

and $F_{S^c}(K)$ follows from

$$K = \sum_{i=1}^{n} K_i.$$
An insurer or financial institution is mostly exposed to $n$ risks: $X_1, X_2, \ldots, X_n$.

Risk measures should be defined for the aggregated risk $S$

$$S = X_1 + X_2 + \ldots + X_n.$$

The following factors will have an impact on the aggregated risk:
- The cdf’s of the marginal risks;
- the dependence between the marginal risks.
Introduction

We consider the problem of deriving upper bounds for risk measures for the aggregate risk with given marginals.

- The cdf’s \( F_{X_1}, F_{X_2}, \ldots, F_{X_n} \) are known;
- the copula connecting the marginal is unknown.
Special case: comonotonicity

- If the cdf’s $F_{X_1}, F_{X_2}, \ldots, F_{X_n}$ are known, we can determine
  \[ \text{VaR}_p[X_i], \text{TVaR}_p[X_i] \text{ and } \mathbb{E}[(X_i - K)_+] \text{ for } i = 1, 2, \ldots, n. \]

- VaR, TVaR are additive for comonotonic risks.
- The stop-loss premium of a comonotonic sum can be decomposed in a sum of marginal stop-loss premiums with appropriate chosen retentions.

**Conclusion:**
- If the risks $X_1, X_2, \ldots, X_n$ are comonotonic, knowledge about the marginal risk measures is sufficient to derive
  \[ \text{VaR}[S^c], \text{TVaR}[S^c] \text{ and } \mathbb{E}[(S^c - K)_+]. \]
Lemma

For any vector \((x_1, x_2, \ldots, x_n)\) and \((K_1, K_2, \ldots, K_n)\) the following inequality holds

\[
\left( \sum_{i=1}^{n} x_i - K \right)_+ \leq \sum_{i=1}^{n} (x_i - K_i)_+ ,
\]

provided \(\sum_{i=1}^{n} K_i = K\).
Assume that $x_i$ is the realization of the loss $X_i$.

$K_i$ is the available capital for loss $X_i$.

$(\sum_{i=1}^{n} x_i - K)_+$ is the shortfall if all risks are merged.

$\sum_{i=1}^{n} (x_i - K_i)_+$ is the shortfall in the stand-alone portfolios.

The inequality shows the effect of diversification.
Proof.

\[ \left( \sum_{i=1}^{n} x_i - K \right)_+ = \left( \sum_{i=1}^{n} (x_i - K_i) \right)_+ \leq \left( \sum_{i=1}^{n} (x_i - K_i)_+ \right)_+ \leq \sum_{i=1}^{n} (x_i - K_i)_+ \]
The comonotonic sum as the convex upper bound

**Theorem**

Consider the random vector \((X_1, X_2, \ldots, X_n)\), its comonotonic modification \((X^c_1, X^c_2, \ldots, X^c_n)\) and the sums \(S = \sum_{i=1}^{n} X_i\) and \(S^c = \sum_{i=1}^{n} X^c_i\). Then

\[
S \preceq_{cx} S^c.
\]

- The least attractive random vector \((X_1, X_2, \ldots, X_n)\) with given marginals \(F_{X_1}, F_{X_2}, \ldots, F_{X_n}\) has the **comonotonic joint distribution**.
- \(S^c\) is a **convex upper bound** for \(S\).
- The stop-loss premium \(\mathbb{E} \left[(S^c - K)_+\right]\) is a convex upper bound for \(\mathbb{E} \left[(S - K)_+\right]\).
Proof

We have the following equality:

$$\mathbb{E}[S] = \mathbb{E}[S^c].$$

Therefore, it suffices to prove stop-loss order:

$$\mathbb{E}[(S - K)_+] \leq \mathbb{E}[(S^c - K)_+] \text{, for all } K \in \left(F_{S^c}^{-1}(1), F_{S^c}^{-1}(0)\right).$$

If $K \notin \left(F_{S^c}^{-1}(0), F_{S^c}^{-1}(1)\right)$: $\mathbb{E}[(S - K)_+] = \mathbb{E}[(S^c - K)_+]$.

If we replace $x_i$ by $X_i$ in Lemma 2 and take expectations, we find for any $K$:

$$\mathbb{E}[(S - K)_+] \leq \sum_{i=1}^{n} \mathbb{E}[(X_i - K_i)_+] ,$$

where $\sum_{i=1}^{n} K_i = K$. 
Proof: Cont’d

- Choose the $K_i$ as in the decomposition formula:

$$K_i = F_{X_i}^{-1}(\alpha_K) (F_{S^c}(K)) .$$

- Then, $\alpha_K$ is chosen such that

$$\sum_{i=1}^{n} K_i = K .$$

- We conclude that for any $K \in \left(F_{S^c}^{-1+}(1), F_{S^c}^{-1}(0)\right)$

$$\mathbb{E} \left[ (S^c - K)_+ \right] = \sum_{i=1}^{n} \mathbb{E} \left[ (X_i - K_i)_+ \right] \geq \mathbb{E} \left[ (S - K)_+ \right] .$$
The following results are already known for long time\(^1\):

- Upper bound for stop-loss premiums of a sum:

\[
\mathbb{E} [(S - K)_+] \leq \mathbb{E} [(S^c - K)_+].
\]

- There exist \(K^*_i\) such that:

\[
\mathbb{E} [(S^c - K)_+] = \sum_{i=1}^{n} \mathbb{E} [(X_i - K^*_i)_+].
\]

Dhaene, Wang, Young, Goovaerts (2000):

- Explicit expression for \(K^*_i\):

\[
K^*_i = F_{X_i}^{-1}(\alpha_K)(F_{S^c}(K)).
\]

\(^1\)see Meilijson & Nádas (1979) and Rüschendorf (2013) for an overview
The convex upper bound as a worst case scenario

**Definition (Fréchet space)**

Fix the marginal cdf’s $F_{X_1}, F_{X_2}, \ldots, F_{X_n}$. The Fréchet space consists of all random vectors having $F_{X_i}$ as marginal cdf’s:

$$
\mathcal{R}_n (F_{X_1}, F_{X_2}, \ldots, F_{X_n}) = \{ \underline{Y} \mid F_{Y_i} \equiv F_{X_i}, \ i = 1, 2, \ldots, n \} .
$$

- Any joint cdf $F_{\underline{Y}}$, with $\underline{Y} \in \mathcal{R}_n$ is a valid candidate to describe the random vector $(X_1, X_2, \ldots, X_n)$.

- Determining actuarial and financial quantities by picking a particular $F_{\underline{Y}}$ can lead to serious misspecification of the true risk.
The convex upper bound as a worst case scenario

**Theorem**

Consider the random vector $\mathbf{X} = (X_1, X_2, \ldots, X_n)$, then we have that

$$\max_{F \in \mathcal{R}_n(F_{X_1}, F_{X_2}, \ldots, F_{X_n})} \mathbb{E}_F [(S - K)_+] = \mathbb{E} [(S^c - K)_+], \quad K \in \mathbb{R}.$$ 

- $\mathbb{E}_F$ is used to denote that the multivariate cdf $F$ is used.

**Interpretation:**

- We know the marginals, but we have no information about the copula;
- this information implies that $F_{\mathbf{X}} \in \mathcal{R}_n (F_{X_1}, F_{X_2}, \ldots, F_{X_n})$;
- $\mathbb{E} [(S^c - K)_+]$ is the largest possible expectation given the available information.
Convex order implies ordered TVaRs:

\[ X \preceq_{cx} Y \iff \begin{cases} 
\mathbb{E}X = \mathbb{E}Y, \\
TVaR_p[X] \leq TVaR_p[Y], \text{ for all } p \in (0,1).
\end{cases} \]

The comonotonic sum \( S^c \) is an convex upper bound for \( S \) (see Theorem 3):

\[ S \preceq_{cx} S^c. \]

We can conclude that

\[ TVaR_p[S] \leq TVaR_p[S^c], \quad p \in (0,1). \]
Assume the marginals $X_i$ are lognormal distributed:

$$X_i \sim LN(\mu_i, \sigma_i^2), \text{ for } i = 1, 2, \ldots, n.$$

Determine the marginal TVaR’s

- If the marginals are lognormal distributed:

$$TVaR_p \left[ F_{X_i}^{-1}(U) \right] = e^{\mu_i + \frac{\sigma_i^2}{2}} \frac{\Phi \left( \sigma_i - \Phi^{-1}(p) \right)}{1 - p}.$$

- In case the marginals have a distribution for which the TVaR is not given in closed form, one can use simulation to determine $TVaR_p \left[ F_{X_i}^{-1}(U) \right]$. 


Illustration: Comonotonicity as a worst case scenario

Tail Value-at-Risk

- $\rho = 30\%$
- $\rho = 50\%$
- $\rho = 80\%$
- Worst case TVaR
3 – Simulating VaR and TVaR

Illustration: Comonotonicity as a worst case scenario

Tail Value-at-Risk

- Gaussian Copula with \( \rho = 0.5 \)
- Clayton copula with \( \theta = 4 \)
- Worst case TVaR
VaR is additive for comonotonic risks:

\[
\text{VaR}_p[S^c] = \sum_{i=1}^{n} \text{VaR}_p \left[ F_{X_i}^{-1}(U) \right].
\]

Determine the marginal VaR’s

- If the marginals are lognormal distributed:

\[
\text{VaR}_p \left[ F_{X_i}^{-1}(U) \right] = e^{\mu_i + \sigma_i \Phi^{-1}(p)}.
\]

- In case the marginals have a distribution for which the VaR is not given in closed form, one can use simulation.

The convex order relation \( S \leq_{cx} S^c \) does not imply that the VaR’s are ordered:

\[
S \leq_{cx} S^c \not\Rightarrow \text{VaR}_p[S] \leq \text{VaR}_p[S^c].
\]
An upper bound for VaR:

\[ \text{VaR}_p[S] \leq \text{TVaR}_p[S]. \]

The bound \( \text{TVaR}_p[S] \) can only be determined if the dependence structure is known.

Another upper bound for VaR:

\[ \text{VaR}_p[S] \leq \sum_{i=1}^{n} \text{TVaR}_p \left( F_{X_i}^{-1}(U) \right). \]

- The upper bound only depends on marginal information;
- but might be too crude...?
3 – Worst case VaR scenarios

Illustration: Comonotonicity does not correspond with the worst-case VaR
The problem of finding a dependence structure for which VaR is maximized is an open problem.


Consider two portfolios with losses $X_1$ and $X_2$.

Solvency capital requirement imposed by the regulator is $\rho$.

**Merged portfolios:**
- The two portfolios are jointly liable for the shortfall of the total loss $X_1 + X_2$.
- Shortfall of the merged portfolio:
  \[(X_1 + X_2 - \rho[X_1 + X_2])_+\].

**Stand-alone portfolios:**
- One portfolio is not liable for the shortfall of the other portfolio.
- Shortfall of the individual portfolios:
  \[(X_i - \rho[X_i])_+ , \quad i = 1, 2.\]
Change in shortfall if we split:

\[(X_1 - \rho[X_1])_+ + (X_2 - \rho[X_2])_+ - (X_1 + X_2 - \rho[X_1 + X_2])_+\].

Consider the following inequality:

\[(X_1 + X_2 - \rho[X_1] - \rho[X_2])_+ \leq (X_1 - \rho[X_1])_+ + (X_2 - \rho[X_2])_+\].

**Conclusion:**

- Assume that \(\rho\) is additive: \(\rho[X_1 + X_2] = \rho[X_1] + \rho[X_2]\).
- Splitting leads to an *increase* of the shortfall.

From the supervisor’s point of view, splitting a merged portfolio leads to a less favorable situation.
Splitting a merged portfolio in two stand-alone portfolios

- If $\rho$ is additive, a split will not affect the total capital requirement.
- Assume that $\rho$ is subadditive:

$$\rho[X_1 + X_2] \leq \rho[X_1] + \rho[X_2].$$

- If the risk measure is subadditive, splitting a merged portfolio will result in a higher (total) capital requirement.

- The shortfall of the merged portfolio:

$$ (X_1 + X_2 - \rho[X_1] - \rho[X_2])_+ \leq (X_1 + X_2 - \rho[X_1 + X_2])_+.$$

Conclusion:

- The shortfall after the split $(X_1 - \rho[X_1])_+ + (X_2 - \rho[X_2])_+$.
- Only a risk measure that is sufficiently subadditive, will guarantee that the split will not increase the shortfall.
Merging two stand-alone portfolios

- We use the following inequality:

\[(X_1 + X_2 - \rho[X_1] - \rho[X_2])_+ \leq (X_1 - \rho[X_1])_+ + (X_2 - \rho[X_2])_+\.

- If \(\rho\) is additive
  - From the regulator’s point of view: merging is desirable.
  - A merger decreases the shortfall risk.

- Capital requirement of the merged portfolio \(= \rho[X_1 + X_2]\).
  - The sum \(\rho[X_1] + \rho[X_2]\) can be smaller than \(\rho[X_1 + X_2]\), without leading to an increase in shortfall.
  - The regulator may allow (to some extent) subadditivity, provided the shortfall risk does not increase.
Consider the losses $X_1$ and $X_2$ of two portfolios.

Available capital is denoted by $K_1$ and $K_2$.

The capital of portfolio $i$ at the end of the reference period is

\[(K_i - X_i)^+ \, .\]

- If the loss $X_i$ is smaller than the capital available, the portfolio is worth $(K_i - X_i)$.
- If the loss $X_i$ exceeds the available capital, the portfolio is ruined.
We have that

\[(K_1 + K_2 - X_1 - X_2) \leq (K_1 - X_1)_+ + (K_2 - X_2)_+,\]

always holds.

Shareholder’s point of view:

- Maximize the end-of-period capital.
- Merging portfolios will lead to a lower end-of-period capital.

Interpretation:

- Merging the portfolios implies that losses in one portfolio can spread to the other portfolio.
- The shareholders will ask for firewalls between the portfolios.
Consider a risk measure $\rho$:

- $\rho$ is law invariant:
  \[ X \overset{d}{=} Y \Rightarrow \rho [X] = \rho [Y] . \]

- $\rho$ is additive for comonotonic risks $X^c$ and $Y^c$:
  \[ \rho [X^c + Y^c] = \rho [X^c] + \rho [Y^c] . \]

- $\rho$ preserves stop-loss order:
  \[ X \preceq_{sl} Y \Rightarrow \rho [X] \leq \rho [Y] . \]

Then $\rho$ is subadditive:

\[ \rho [X + Y] \leq \rho [X] + \rho [Y] \]

Exercise: Prove this theorem.
The TVaR is law invariant, preserves stop-loss order and is additive for comonotonic risks:

TVaR is a subadditive risk measure:

\[ \text{TVaR}_p[X + Y] \leq \text{TVaR}_p[X] + \text{TVaR}_p[Y]. \]

TVaR is also monotonic, positive homogeneous and translation invariant:

TVaR is a coherent measure of risk.
Consider the Bernoulli r.v.’s X and Y. We assume that $X \overset{d}{=} Y$ and

$$P[X = 0] = 0.98 = 1 - P[X = 1].$$

VaR of X and Y:

$$VaR_{0.975}[X] = VaR_{0.975}[Y] = 0.$$

Assume that X and Y are independent:

$$P[X + Y = 0] = 0.9604.$$

We conclude that:

$$VaR_{0.975}[X + Y] > VaR_{0.975}[X] + VaR_{0.975}[Y] = 0.$$
Applying a subadditive risk measure in a merger, can increase the shortfall.

- In this case, the risk measure is *too subadditive*.
- The regulator needs to restrict the benefit one can obtain by merging stand-alone portfolios.

A solvency requirement $\rho$ should satisfy the following relation:

$$\mathbb{E} [(X_1 + X_2 - \rho[X_1 + X_2])_+ ] \leq \sum_{i=1}^{2} \mathbb{E} [(X_i - \rho[X_i])_+] . \quad (1)$$

**Theorem:**

- Consider a law invariant, translation invariant and positive homogeneous risk measure $\rho$.
- Assume that $X_1$, $X_2$ and $X_1 + X_2$ belong to the same location-scale family of distributions and have finite variances.
- Then, equation (1) is fulfilled.
This theorem holds for a wide range of elliptical distributions.

- The multivariate normal distribution is an example.

Under the conditions of the theorem: ‘the hunger for subadditivity is never satisfied.’

- Splitting two merged portfolios will always lead to an increase of the expected shortfall.

TVaR is an example of a risk measure which can be too subadditive.
Consider the following distribution:

\[ X_1 \sim \text{Uniform}(0, 1) \]

\[ X_2 = \begin{cases} 
0.9U, & \text{if } 0 < X_1 \leq 0.9 \\
X_1, & \text{if } 0.9 < X_1 \leq 1.
\end{cases} \]

\( U \) is independent from \( X_1 \).

\( X_1 \) and \( X_2 \) are uniform(0, 1) distributed and for \( p = 0.85 \) we find:

\[ \text{TVaR}_p[X_j] = \frac{1 + p}{2} = 0.925, \]

and

\[ \mathbb{E} \left[ (X_j - \text{TVaR}_p[X_j])_+ \right] = \frac{(1 - p)^2}{8} = 0.0028125. \]
Consider the sum \( S = X_1 + X_2 \).

The distribution function \( F_S \) is given by the following expression:

\[
F_S(s) = \begin{cases} 
\frac{s^2}{1.8}, & 0 < s \leq 0.9, \\
-\frac{s^2}{1.8} + 2s - 0.9, & 0.9 < s \leq 1.8, \\
\frac{s}{2}, & 1.8 < s < 2.
\end{cases}
\]

The cdf \( F_S \) is strictly increasing. For \( p = 0.85 \), \( \text{VaR}_p[S] \) can be determined as follows:

\[
F_S^{-1}(p) = x
\]

\[\Leftrightarrow \quad F_S(x) = p
\]

\[\Leftrightarrow \quad 2x - \frac{x^2}{1.8} - 0.9 = p.
\]

If we solve this quadratic equation in \( x \), we find that

\[F_S^{-1}(0.85) = 1.5.\]
For $K \in (0.9, 1.8)$, we have that $\mathbb{E} [(S - K)_+]$ is given by

$$\mathbb{E} [(S - K)_+] = \int_{K}^{2} (1 - F_S(y)) \, dy$$

$$= \int_{K}^{1.8} (1 - F_S(y)) \, dy + \int_{1.8}^{2} (1 - F_S(y)) \, dy$$

$$= -1.9K + K^2 - \frac{K^3}{5.4} + 1.27.$$

The TVaR can be calculated using

$$\text{TVaR}_p[S] = \text{VaR}_p[S] + \frac{1}{1 - p} \mathbb{E} \left[ (S - \text{VaR}_p[S])_+ \right],$$

which results in

$$\text{TVaR}_{0.85}[S] \approx 1.8.$$
6 – Can a risk measure be too subadditive?

TVaR can be too subadditive: an example

- Expected shortfall of the merged portfolio with \( p = 0.85 \):
  \[
  \mathbb{E} \left[ (S - \operatorname{TVaR}_p[S])_+ \right] = 0.01.
  \]

- Expected shortfall of the two stand alone portfolios:
  \[
  \mathbb{E} \left[ (X_1 - \operatorname{TVaR}_p[X_1])_+ \right] + \mathbb{E} \left[ (X_2 - \operatorname{TVaR}_p[X_2])_+ \right] = 0.006.
  \]

- TVaR is too subadditive:
  - When the risks \( X_1 \) and \( X_2 \) are merged, the risk of insolvency is larger than in a situation where these risks are considered separately.