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Foundations of Quantitative Risk Measurement

Subadditivity and risk aggregation

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December, 2019

- Consider the risks $X_1, X_2, \ldots, X_n$.
- The *aggregated risk* is denoted by S:

$$S = X_1 + X_2 + \ldots + X_n.$$

- Knowledge of the cdf F_S is required when determining risk measures for the aggregated risk S.
    - The cdf F_S is determined by
        - Marginal information: $F_{X_1}, F_{X_2}, \ldots, F_{X_n}$.
        - Dependence information: The copula C connects the marginals.

Introduction

- Estimating the cdf F_S
  - estimating the marginal cdfs is possible and can, often, be done accurately.
  - estimaing the copula is harder, especially when n is large.
- In general, the cdf F_S will be unknown or too cumbersome to work with.
  - Determining $\mathrm{TVaR}_p[S]$, $\mathrm{VaR}_p[S]$, $\mathbb{E}\left[(S-K)_+\right]$ is not an easy task.
- In case we have comonotonic risks, it is possible to determine $\mathrm{TVaR}_p[S^c]$, $\mathrm{VaR}_p[S^c]$, $\mathbb{E}\left[(S^c-K)_+\right]$.

TVaR and α-inverses

- Recall that TVaR can be written as:

$$\mathsf{TVaR}_p\left[X\right] = F_X^{-1}\left(p\right) + \frac{1}{1-p}\mathbb{E}\left[\left(X - F_X^{-1}(p)\right)_+\right].$$

- Take $p \in (0,1)$. The stop-loss premium can be rewritten:

$$\begin{aligned}\mathbb{E}\left[\left(X - F_X^{-1(\alpha)}(p)\right)_+\right] &= \mathbb{E}\left[\left(X - F_X^{-1}(p)\right)_+\right] \\ &\quad - \left(F_X^{-1(\alpha)}(p) - F_X^{-1}(p)\right)(1-p)\end{aligned}$$

- Combining the two equations results in

$$\mathsf{TVaR}_p\left[X\right] = F_X^{-1(\alpha)}\left(p\right) + \frac{1}{1-p}\mathbb{E}\left[\left(X - F_X^{-1(\alpha)}(p)\right)_+\right].$$

- Consider the comonotonic sum S^c

$$S^c = F_{X_1}^{-1}(U) + F_{X_2}^{-1}(U) + \ldots F_{X_n}^{-1}(U).$$

- TVaR is additive for comonotonic risks:

$$\mathsf{TVaR}_p[S^c] = \mathsf{TVaR}_p[X_1] + \mathsf{TVaR}_p[X_2] + \ldots + \mathsf{TVaR}_p[X_n].$$

- Using the expression of TVaR in terms of alpha inverses, we find:

$$\mathbb{E}\left[\left(S^c - F_{S^c}^{-1(\alpha)}(p)\right)_+\right] = \sum_{i=1}^{n} \mathbb{E}\left[\left(X_i - F_{X_i}^{-1(\alpha)}(p)\right)_+\right].$$

Decomposition formula

- Choose a $K \in \left(F_{S^c}^{-1+}(0), F_{S^c}^{-1}(1)\right)$. Then there always exists an α_K such that
$$K = F_{S^c}^{-1(\alpha_K)}\left(F_{S^c}(K)\right).$$
- Take $p = F_{S^c}(K)$, then we find
$$\mathbb{E}\left[(S^c - K)_+\right] = \sum_{i=1}^{n} \mathbb{E}\left[\left(X_i - F_{X_i}^{-1(\alpha_K)}(p)\right)_+\right].$$
- If $K \neq \left(F_{S^c}^{-1+}(0), F_{S^c}^{-1}(1)\right)$:
  - $K \leq F_{S^c}^{-1+}(0):\ \mathbb{E}\left[(S^c - K)_+\right] = \mathbb{E}[S] - K.$
  - $K \geq F_{S^c}^{-1}(1):\ \mathbb{E}\left[(S^c - K)_+\right] = 0.$

Decomposition formula

Theorem

For $K \in \left(F_{S^c}^{-1+}(0), F_{S^c}^{-1}(1)\right)$, *the stop-loss premiums* $\mathbb{E}\left[(S^c - K)_+\right]$ *can be written as*

$$\mathbb{E}\left[(S^c - K)_+\right] = \sum_{i=1}^{n} \mathbb{E}\left[(X_i - K_i)_+\right],$$

where

$$K_i = F_{X_i}^{-1(\alpha_K)}\left(F_{S^c}(K)\right),$$

and α_K *follows from*

$$K = \sum_{i=1}^{n} K_i.$$

- We can prove a stronger result:

$$\left(\sum_{i=1}^{n} F_{X_i}^{-1}(U) - K\right)_+ = \sum_{i=1}^{n} \left(F_{X_i}^{-1}(U) - K_i\right)_+.$$

Decomposition formula for strictly increasing marginals

- Assume that each cdf F_{X_i} is strictly increasing.
- Then, also F_{S^c} is strictly increasing:

$$F_{S^c}^{-1(\alpha_K)}\left(F_{S^c}(K)\right) = F_{S^c}^{-1}\left(F_{S^c}(K)\right).$$

- For $K \in \left(F_{S^c}^{-1+}(0), F_{S^c}^{-1}(1)\right)$, we find

$$\mathbb{E}\left[(S^c - K)_+\right] = \sum_{i=1}^{n} \mathbb{E}\left[(X_i - K_i)_+\right],$$

where

$$K_i = F_{X_i}^{-1}\left(F_{S^c}(K)\right),$$

and $F_{S^c}(K)$ follows from

$$K = \sum_{i=1}^{n} K_i.$$

Introduction

- An insurer or financial institution is mostly exposed to n risks: $X_1, X_2, \ldots, X_n$.
- Risk measures should be defined for the aggregated risk S

$$S = X_1 + X_2 + \ldots + X_n.$$

- The following factors will have an impact on the aggregated risk:
  - The cdf's of the marginal risks;
  - the dependence between the marginal risks.

Introduction

- We consider the problem of deriving upper bounds for risk measures for the

  aggregate risk with given marginals.

  - The cdf's $F_{X_1}, F_{X_2}, \ldots, F_{X_n}$ are known;
  - the copula connecting the marginal is unknown.

Special case: comonotonicity

- If the cdf's $F_{X_1}, F_{X_2}, \ldots, F_{X_n}$ are known, we can determine

$$\text{VaR}_p[X_i],\ \text{TVaR}_p[X_i] \text{ and } \mathbb{E}\left[(X_i - K)_+\right] \text{ for } i = 1, 2, \ldots, n.$$

  - VaR, TVaR are additive for comonotonic risks.
  - The stop-loss premium of a comonotonic sum can be decomposed in a sum of marginal stop-loss premiums with appropriate chosen retentions.

- Conclusion:
  - If the risks $X_1, X_2, \ldots, X_n$ are **comonotonic**, knowledge about the marginal risk measures is sufficient to derive

$$\text{VaR}[S^c],\ \text{TVaR}[S^c] \text{ and } \mathbb{E}\left[(S^c - K)_+\right].$$

Lemma

For any vector $(x_1, x_2, \ldots, x_n)$ *and* $(K_1, K_2, \ldots, K_n)$ *the following inequality holds*

$$\left(\sum_{i=1}^{n} x_i - K \right)_+ \leq \sum_{i=1}^{n} (x_i - K_i)_+ ,$$

provided $\sum_{i=1}^{n} K_i = K$.

- Assume that x_i is the realization of the loss X_i.
- K_i is the available capital for loss X_i.
- $\left(\sum_{i=1}^{n} x_i - K\right)_+$ is the shortfall if all risks are *merged*.
- $\sum_{i=1}^{n} \left(x_i - K_i\right)_+$ is the shortfall in the stand-alone portfolios.
- The inequality shows the effect of <u>diversification</u>.

Proof.

$$\begin{aligned}\left(\sum_{i=1}^{n} x_i - K\right)_+ &= \left(\sum_{i=1}^{n} (x_i - K_i)\right)_+ \\ &\leq \left(\sum_{i=1}^{n} (x_i - K_i)_+\right)_+ \\ &\leq \sum_{i=1}^{n} (x_i - K_i)_+\end{aligned}$$

□

The comonotonic sum as the convex upper bound

Theorem

Consider the random vector $(X_1, X_2, \ldots, X_n)$, its comonotonic modification $(X_1^c, X_2^c, \ldots, X_n^c)$ and the sums $S = \sum_{i=1}^n X_i$ and $S^c = \sum_{i=1}^n X_i^c$. Then

$$S \preceq_{cx} S^c.$$

- The least attractive random vector $(X_1, X_2, \ldots, X_n)$ with given marginals $F_{X_1}, F_{X_2}, \ldots, F_{X_n}$ has the **comonotonic joint distribution**.
- S^c is a *convex upper bound* for S.
- The stop-loss premium $\mathbb{E}\left[(S^c - K)_+\right]$ is a convex upper bound for $\mathbb{E}\left[(S - K)_+\right]$

Proof

- We have the following equality:

$$\mathbb{E}[S] = \mathbb{E}[S^c].$$

- Therefore, it suffices to prove **stop-loss order**:

$$\mathbb{E}\left[(S-K)_+\right] \leq \mathbb{E}\left[(S^c-K)_+\right], \text{ for all } K \in \left(F_{S^c}^{-1+}(1), F_{S^c}^{-1}(0)\right).$$

- If $K \notin \left(F_{S^c}^{-1+}(0), F_{S^c}^{-1}(1)\right)$: $\mathbb{E}\left[(S-K)_+\right] = \mathbb{E}\left[(S^c-K)_+\right]$.
- If we replace x_i by X_i in Lemma 2 and take expectations, we find for any K:

$$\mathbb{E}\left[(S-K)_+\right] \leq \sum_{i=1}^{n} \mathbb{E}\left[(X_i-K_i)_+\right],$$

where $\sum_{i=1}^{n} K_i = K$.

Proof: Cont'd

- Choose the K_i as in the decomposition formula:

$$K_i = F_{X_i}^{-1(\alpha_K)}\left(F_{S^c}(K)\right).$$

- Then, α_K is chosen such that

$$\sum_{i=1}^{n} K_i = K$$

.

- We conclude that for any $K \in \left(F_{S^c}^{-1+}(1), F_{S^c}^{-1}(0)\right)$

$$\begin{aligned}\mathbb{E}\left[(S^c - K)_+\right] &= \sum_{i=1}^{n} \mathbb{E}\left[(X_i - K_i)_+\right] \\ &\geq \mathbb{E}\left[(S - K)_+\right].\end{aligned}$$

- The following results are already known for long time[1]:
  - Upper bound for stop-loss premiums of a sum:

$$\mathbb{E}\left[(S-K)_+\right] \quad \leq \quad \mathbb{E}\left[(S^c-K)_+\right].$$

  - There exist K_i^* such that:

$$\mathbb{E}\left[(S^c-K)_+\right] = \sum_{i=1}^{n} \mathbb{E}\left[(X_i-K_i^*)_+\right].$$

- Dhaene, Wang, Young, Goovaerts (2000):
  - Explicit expression for K_i^*:

$$K_i^* = F_{X_i}^{-1(\alpha_K)}\left(F_{S^c}(K)\right).$$

[1]see Meilijson & Nádas (1979) and Rüschendorf (2013) for an overview

The convex upper bound as a worst case scenario

Definition (Fréchet space)

Fix the marginal cdf's $F_{X_1}, F_{X_2}, \ldots, F_{X_n}$. The Fréchet space consists of all random vectors having F_{X_i} as marginal cdf's:

$$\mathcal{R}_n \left(F_{X_1}, F_{X_2}, \ldots, F_{X_n} \right) = \left\{ \underline{Y} \mid F_{Y_i} \equiv F_{X_i},\ i = 1, 2, \ldots, n \right\}.$$

- Any joint cdf $F_{\underline{Y}}$, with $\underline{Y} \in \mathcal{R}_n$ is a valid candidate to describe the random vector $(X_1, X_2, \ldots., X_n)$.
- Determining actuarial and financial quantities by picking a particular $F_{\underline{Y}}$ can lead to serious misspecification of the true risk.

The convex upper bound as a worst case scenario

Theorem

Consider the random vector $\underline{X} = (X_1, X_2, \ldots, X_n)$, *then we have that*

$$\max_{F \in \mathcal{R}_n\left(F_{X_1}, F_{X_2}, \ldots, F_{X_n}\right)} \mathbb{E}_F\left[(S-K)_+\right] = \mathbb{E}\left[(S^c-K)_+\right], \quad K \in \mathbb{R}.$$

- $\mathbb{E}_F$ is used to denote that the multivariate cdf F is used.
- Interpretation:
  - We know the marginals, but we have no information about the copula;
  - this information implies that $F_{\underline{X}} \in \mathcal{R}_n\left(F_{X_1}, F_{X_2}, \ldots, F_{X_n}\right)$;
  - $\mathbb{E}\left[(S^c-K)_+\right]$ is the largest possible expectation given the available information

- Convex order implies ordered TVaRs:

$$X \preceq_{cx} Y \Longleftrightarrow \begin{cases} \mathbb{E}X = \mathbb{E}Y, \\ TVaR_p[X] \leq TVaR_p[Y], \text{ for all } p \in (0,1). \end{cases}$$

- The comonotonic sum S^c is an convex upper bound for S (see Theorem 3):

$$S \preceq_{cx} S^c.$$

- We can conclude that

$$\text{TVaR}_p[S] \leq \text{TVaR}_p[S^c], \quad p \in (0,1).$$

Lognormal marginals

- Assume the marginals X_i are lognormal distributed:

$$X_i \overset{d}{=} LN(\mu_i, \sigma_i^2), \text{ for } i = 1, 2, \ldots, n.$$

- Determine the marginal TVaR's
  - If the marginals are lognormal distributed:

$$\text{TVaR}_p\left[F_{X_i}^{-1}(U)\right] = e^{\mu_i + \frac{\sigma_i^2}{2}} \frac{\Phi\left(\sigma_i - \Phi^{-1}(p)\right)}{1-p}.$$

  - In case the marginals have a distribution for which the TVaR is not given in closed form, one can use simulation to determine $\text{TVaR}_p\left[F_{X_i}^{-1}(U)\right]$.

Illustration: Comonotonicity as a worst case scenario

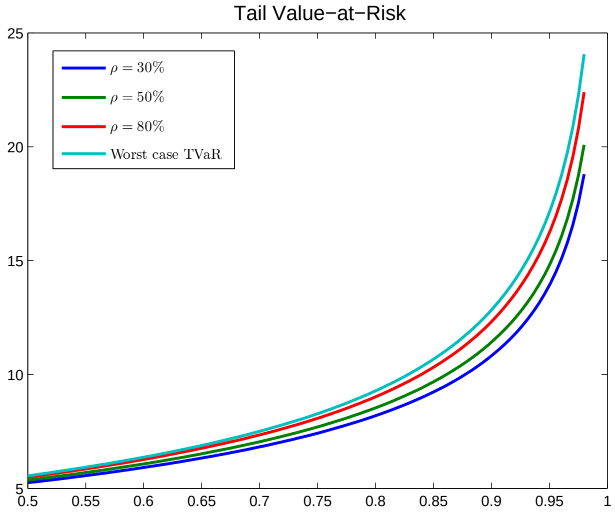
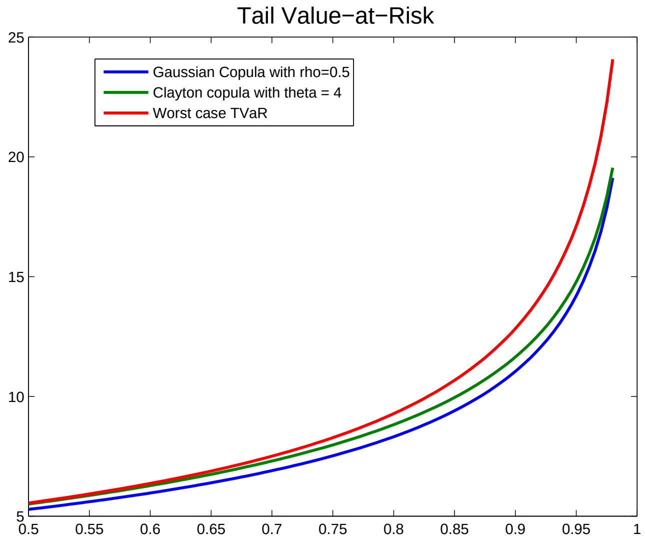


Illustration: Comonotonicity as a worst case scenario



Comonotonicity and the worst-case VaR

- VaR is additive for comonotonic risks:

$$\text{VaR}_p[S^c] = \sum_{i=1}^{n} \text{VaR}_p\left[F_{X_i}^{-1}(U)\right].$$

- Determine the marginal VaR's
  - If the marginals are lognormal distributed:

$$\text{VaR}_p\left[F_{X_i}^{-1}(U)\right] = e^{\mu_i + \sigma_i \Phi^{-1}(p)}.$$

  - In case the marginals have a distribution for which the VaR is not given in closed form, one can use simulation.

- The convex order relation $S \preceq_{cx} S^c$ does not imply that the VaR's are ordered:

$$S \preceq_{cx} S^c \nRightarrow \text{VaR}_p[S] \leq \text{VaR}_p[S^c].$$

Comonotonicity and the worst-case VaR

- An upper bound for VaR:

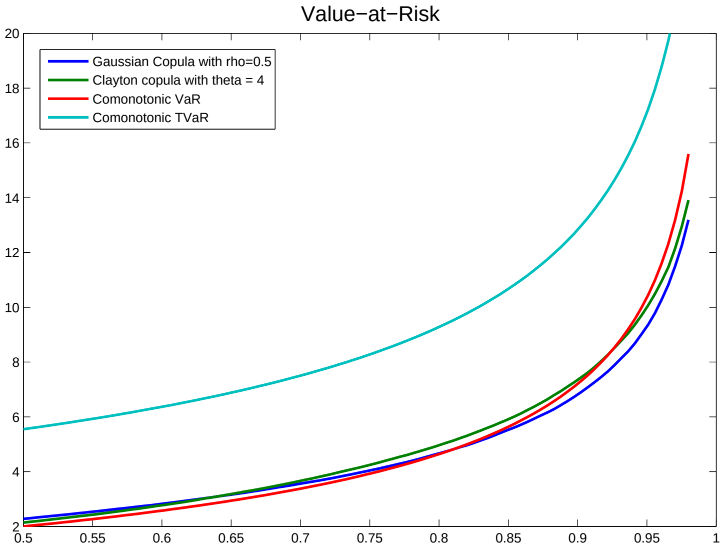
$$\mathrm{VaR}_p[S] \leq \mathrm{TVaR}_p[S].$$

- The bound $\mathrm{TVaR}_p[S]$ can only be determined if the dependence structure is known.
- Another upper bound for VaR:

$$\mathrm{VaR}_p[S] \leq \sum_{i=1}^{n} \mathrm{TVaR}_p\left[F_{X_i}^{-1}(U)\right].$$

  - The upper bound only depends on marginal information;
  - but might be too crude...?

Illustration: Comonotonicity does not correspond with the worst-case VaR



Determining upper bounds for VaR

- The problem of finding a dependence structure for which VaR is maximized is an open problem.
- Makarov (1981) solved the problem for $n = 2$.
- Denuit, Genest & Marceau (1999) and Embrechts & Puccetti (2006) derive bounds for $F_S(x)$ using stochastic dominance.
- Embrechts, Puccetti & Rüschendorf (2013) and Wang, Peng & Yang (2013) consider worst case VaR scenarios.

- Consider two portfolios with losses X_1 and X_2.
- Solvency capital requirement imposed by the regulator is ρ.
- Merged portfolios:
  - The two portfolios are jointly liable for the shortfall of the total loss $X_1 + X_2$.
  - Shortfall of the merged portfolio:

$$(X_1 + X_2 - \rho[X_1 + X_2])_+ .$$

- Stand-alone portfolios:
  - One portfolio is not liable for the shortfall of the other portfolio.
  - Shortfall of the individual portfolios:

$$(X_i - \rho[X_i])_+ , \quad i = 1, 2.$$

Splitting a merged portfolio in two stand-alone portfolios

- Change in shortfall if we split:

$$(X_1 - \rho[X_1])_+ + (X_2 - \rho[X_2])_+ - (X_1 + X_2 - \rho[X_1 + X_2])_+ .$$

- Consider the following inequality:

$$(X_1 + X_2 - \rho[X_1] - \rho[X_2])_+ \leq (X_1 - \rho[X_1])_+ + (X_2 - \rho[X_2])_+ .$$

- Conclusion:
  - Assume that ρ is additive: $\rho[X_1 + X_2] = \rho[X_1] + \rho[X_2]$.
  - Splitting leads to an *increase* of the shortfall.
- From the supervisor's point of view, splitting a merged portfolio leads to a less favorable situation.

Splitting a merged portfolio in two stand-alone portfolios

- If ρ is additive, a split will not affect the total capital requirement.
- Assume that ρ is *subadditive*:

$$\rho[X_1 + X_2] \leq \rho[X_1] + \rho[X_2].$$

- If the risk measure is subadditive, splitting a merged portfolio will result in a higher (total) capital requirement.
- The shortfall of the merged portfolio:

$$(X_1 + X_2 - \rho[X_1] - \rho[X_2])_+ \leq (X_1 + X_2 - \rho[X_1 + X_2])_+ .$$

- Conclusion:
  - The shortfall after the split $(X_1 - \rho[X_1])_+ + (X_2 - \rho[X_2])_+$.
  - Only a risk measure that is sufficiently subadditive, will guarantee that the split will not increase the shortfall.

Merging two stand-alone portfolios

- We use the following inequality:

$$(X_1 + X_2 - \rho[X_1] - \rho[X_2])_+ \leq (X_1 - \rho[X_1])_+ + (X_2 - \rho[X_2])_+ .$$

- If ρ is additive
  - From the regulator's point of view: merging is desirable.
  - A merger decreases the shortfall risk.
- Capital requirement of the merged portfolio $= \rho[X_1 + X_2]$.
  - The sum $\rho[X_1] + \rho[X_2]$ can be smaller than $\rho[X_1 + X_2]$, without leading to an increase in shortfall.
  - The regulator may allow (to some extent) subadditivity, provided the shortfall risk does not increase.

Diversification from the viewpoint of the shareholder

- Consider the losses X_1 and X_2 of two portfolios.
- Available capital is denoted by K_1 and K_2.
- The capital of portfolio i at the end of the reference period is

$$(K_i - X_i)_+.$$

  - If the loss X_i is smaller than the capital available, the portfolio is worth

$$(K_i - X_i).$$

  - If the loss X_i exceeds the available capital, the portfolio is ruined.

Diversification from the viewpoint of the shareholder

- We have that

$$(K_1 + K_2 - X_1 - X_2) \leq (K_1 - X_1)_+ + (K_2 - X_2)_+,$$

  always holds.

- Shareholder's point of view:
  - Maximize the end-of-period capital.
  - merging portfolios will lead to a lower end-of-period capital.
- Interpretation:
  - Merging the portfolios implies that losses in one portfolio can spread to the other portfolio.
  - The shareholders will ask for *firewalls* between the portfolios.

Sufficient conditions for subadditivity

- Consider a risk measure ρ:
  - ρ is law invariant:
$$X \stackrel{d}{=} Y \Rightarrow \rho[X] = \rho[Y].$$
  - ρ is additive for comonotonic risks X^c and Y^c:
$$\rho[X^c + Y^c] = \rho[X^c] + \rho[Y^c].$$
  - ρ preserves stop-loss order:
$$X \preceq_{sl} Y \Rightarrow \rho[X] \leq \rho[Y].$$
- Then ρ is subadditive:
$$\rho[X+Y] \leq \rho[X] + \rho[Y]$$
- Exercise: Prove this theorem.

- The TVaR is law invariant, preserves stop-loss order and is additive for comonotonic risks:
- TVaR is a **subadditive** risk measure:

$$\mathsf{TVaR}_p[X+Y] \leq \mathsf{TVaR}_p[X] + \mathsf{TVaR}_p[Y].$$

- TVaR is also monotonic, positive homogeneous and translation invariant:
- TVaR is a **coherent** measure of risk.

VaR is not subadditive: an example

- Consider the Bernoulli r.v.'s X and Y. We assume that $X \stackrel{d}{=} Y$ and

$$\mathbb{P}[X = 0] = 0.98 = 1 - \mathbb{P}[X = 1].$$

- VaR of X and Y:

$$VaR_{0.975}[X] = VaR_{0.975}[Y] = 0.$$

- Assume that X and Y are independent:

$$\mathbb{P}[X + Y = 0] = 0.9604.$$

- We conclude that:

$$VaR_{0.975}[X + Y] > VaR_{0.975}[X] + VaR_{0.975}[Y] = 0.$$

- Applying a subadditive risk measure in a merger, can increase the shortfall.
  - In this case, the risk measure is *too subadditive*.
  - The regulator needs to restrict the benefit one can obtain by merging stand-alone portfolios.
- A solvency requirement ρ should satisfy the following relation:

$$\mathbb{E}\left[(X_1 + X_2 - \rho[X_1 + X_2])_+\right] \leq \sum_{i=1}^{2} \mathbb{E}\left[(X_i - \rho[X_i])_+\right]. \tag{1}$$

- <u>Theorem:</u>
  - Consider a law invariant, translation invariant and positive homogeneous risk measure ρ.
  - Assume that X_1, X_2 and $X_1 + X_2$ belong to the same location-scale family of distributions and have finite variances.
  - Then, equation (1) is fulfilled.

- This theorem holds for a wide range of elliptical distributions.
  - The multivariate normal distribution is an example.
- Under the conditions of the theorem: '*the hunger for subadditivity is never satisfied.*'
  - Splitting two merged portfolios will always lead to an increase of the expected shortfall.
- TVaR is an example of a risk measure which can be too subadditive.

TVaR can be too subadditive: an example

- Consider the following distribution:

$$\begin{aligned} X_1 &\sim \text{Uniform}\,(0,1) \\ X_2 &= \begin{cases} 0.9U, & \text{if } 0 < X_1 \leq 0.9 \\ X_1, & \text{if } 0.9 < X_1 \leq 1. \end{cases} \end{aligned}$$

- U is independent from X_1.
- X_1 and X_2 are uniform$(0,1)$ distributed and for $p = 0.85$ we find:

$$\text{TVaR}_p[X_j] = \frac{1+p}{2} = 0.925,$$

and

$$\mathbb{E}\left[\left(X_j - \text{TVaR}_p[X_j]\right)_+\right] = \frac{(1-p)^2}{8} = 0.0028125.$$

TVaR can be too subadditive: an example

- Consider the sum $S = X_1 + X_2$.
- The distribution function F_S is given by the following expression

$$F_S(s) = \begin{cases} \frac{s^2}{1.8}, & 0 < s \leq 0.9, \\ -\frac{s^2}{1.8} + 2s - 0.9, & 0.9 < s \leq 1.8, \\ \frac{s}{2}, & 1.8 < s < 2. \end{cases}$$

- The cdf F_S is strictly increasing. For $p = 0.85$, $\text{VaR}_p[S]$ can be determined as follows:

$$\begin{aligned} & F_S^{-1}(p) = x \\ \Leftrightarrow\ & F_S(x) = p \\ \Leftrightarrow\ & 2x - \frac{x^2}{1.8} - 0.9 = p. \end{aligned}$$

If we solve this quadratic equation in x, we find that

$$F_S^{-1}(0.85) = 1.5.$$

TVaR can be too subadditive: an example

- For $K \in (0.9, 1.8)$, we have that $\mathbb{E}\left[(S-K)_+\right]$ is given by

$$\begin{aligned}\mathbb{E}\left[(S-K)_+\right] &= \int_K^2 (1-F_S(y))\,\mathrm{d}y \\ &= \int_K^{1.8} (1-F_S(y))\,\mathrm{d}y + \int_{1.8}^2 (1-F_S(y))\,\mathrm{d}y \\ &= -1.9K + K^2 - \frac{K^3}{5.4} + 1.27.\end{aligned}$$

- The TVaR can be calculated using

$$\mathrm{TVaR}_p[S] = \mathrm{VaR}_p[S] + \frac{1}{1-p}\mathbb{E}\left[\left(S - \mathrm{VaR}_p[S]\right)_+\right],$$

which results in

$$\mathrm{TVaR}_{0.85}[S] \approx 1.8.$$

TVaR can be too subadditive: an example

- Expected shortfall of the merged portfolio with $p = 0.85$:

$$\mathbb{E}\left[\left(S - \mathsf{TVaR}_p[S]\right)_+\right] = 0.01.$$

- Expected shortfall of the two stand alone portfolios:

$$\mathbb{E}\left[\left(X_1 - \mathsf{TVaR}_p[X_1]\right)_+\right] + \mathbb{E}\left[\left(X_2 - \mathsf{TVaR}_p[X_2]\right)_+\right] = 0.006.$$

- TVaR is too subadditive:
  - When the risks X_1 and X_2 are merged, the risk of insolvency is larger than in a situation where these risks are considered separately.